Math 4575 : HW # 1

All problems are from Introductory Combinatorics (5th edition), Chapter 2.
1, 2, 4, 11, 38, 39, 41, 54, 56, 57, 58, 63

- #1
  For each of the four subsets of the two properties (a) and (b), count the number of four-digit numbers whose digits are either 1,2,3,4, or 5.
  (a) The digits are distinct. (b) The number is even.

  - (No restriction) 5^4
  - (Digits are distinct) $5 \times 4 \times 3 \times 2$
  - (Even) $5^3 \times 2$. (The last digit has to be 2, but otherwise there is no restriction.)
  - (Digits are distinct and even) $2 \times 4 \times 3 \times 2 \times 1$. (Choose the digits from right to left.)

- #2
  $$4! \times (13!)^4$$
  (Order the suits, then order the cards within each suit.)

- #4
  How many distinct positive divisors does each of the following numbers have?
  (a) $3^4 \times 5^2 \times 7^6 \times 11^1$
  $$5 \times 3 \times 7 \times 2 = 210.$$
  (b) 620
  Factoring into primes, $620 = 2^2 \times 5^1 \times 31^1$, so there are
  $$3 \times 2 \times 2 = 12$$
  distinct positive divisors.

  (c) 10^{10}
  Since $10^{10} = 2^{10} \times 5^{10}$, there are
  $$11 \times 11 = 121$$
  distinct positive divisors.

- #11
  How many sets of three integers between 1 and 20 are possible if no two consecutive integers are to be in a set?
  Think of placing 3 numbers in the "gaps" between the other 17. There are 18 gaps including the two ends, so there are

  $$\binom{18}{3} = 816$$

  sets of three integers possible.

- #38
  How many integral solutions of $x_1 + x_2 + x_3 + x_4 = 30$ satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, $x_5 \geq 8$?

  We make the substitution $y_1 = x_1 - 2$, $y_2 = x_2$, $y_3 = x_3 + 5$, $y_4 = x_4 - 8$. Then we are counting solutions to $y_1 + y_2 + y_3 + y_4 = 25$, with $y_i \geq 0$ for $1 \leq i \leq 4$. By "stars and bars" counting, the number of solutions is

  $$\binom{28}{3} = 3276.$$

- #39
  There are 20 identical sticks lined up in a row occupying 20 distinct places as follows. Six of them are to be chosen.

  - How many choices are there?

    $$\binom{20}{6} = 38760$$

  - How many choices are there if no two of the chosen sticks can be consecutive?
    Think of 15 gaps between 14 sticks, so

    $$\binom{15}{6} = 5005.$$

    Alternately, by thinking of the chosen sticks as partitioning the non-chosen 14 sticks into 7 parts, this is equivalent to counting solutions to $x_1 + x_2 + \cdots + x_7 = 14$, with $x_1 \geq 0$, $x_i \geq 1$ for $1 \leq i \leq 5$, and $x_7 \geq 0$. (The end parts only need to be non-negative, since the leftmost or rightmost part might have zero sticks, without interfering with the condition that no two sticks can be consecutive.) After a change of variables, stars and bars counting gives the same answer.

– How many choices are there if there must be at least two sticks between each pair of chosen sticks?
$x_1 + x_2 + \cdots + x_7 = 14$, with $x_1 \geq 0$, $x_i \geq 2$ for $1 \leq i \leq 6$, $x_7 \geq 0$. Set $y_1 = x_1$, $y_i = x_i - 2$ for $1 \leq i \leq 6$, $y_7 \geq x_7$. Then $y_1 + \cdots + y_7 = 4$ with $y_i \geq 0$ for $i = 1, \ldots, 7$. By stars and bars counting, there are

$$\binom{10}{6} = 210$$

ways to do this.

- #41
In how many ways can 12 indistinguishable apples and 1 orange be distributed among three children in such a way that each child gets at least one piece of fruit?
Number the children 1, 2, 3. First, choose who gets the orange. Suppose it is child 1. Then the apples need be partitioned among the three children such that 1 receives a non-negative number, and children 2 and 3 each receive at least one. So we are counting solutions to

$$x_1 + x_2 + x_3 = 12$$

with $x_1 \geq 0$, $x_2 \geq 1$, and $x_3 \geq 1$. After a change of variables we have

$$y_1 + y_2 + y_3 = 10$$

with non-negative y_i. There are $\binom{12}{2}$ solutions to this equation. The computation would be the same if the orange was given to child 2 or child 3. So, in total, there are

$$3 \times \binom{12}{2}$$

ways to distribute the fruit.

- #54
Determine the number of towers of the form

$$\emptyset \subseteq A \subseteq B \subseteq \{1, 2, ..., n\}.$$

  – Solution #1. In every tower, we consider whether an element is in both A and B, in neither, or in B only. (The only possibility that is not allowed is being in A only—if an element is contained in A but not B, then A is not contained in B.) We can make this choice independently for each element, so there are 3^n possibilities total.
  – Solution #2. Suppose that B has exactly k elements. Then are $\binom{n}{k}$ ways to choose B, then 2^k possibilities for A. Letting k vary from 0 to n, we arrive at the summation formula

$$\sum_{k=0}^{n} \binom{n}{k} 2^k.$$

The binomial theorem gives that

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n,$$

so substituting $x = 2$ and $y = 1$,

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n.$$

- #56
What is the probability that a poker hand contains a flush (that is, five cards of the same suit)?
Choose the suit, then choose five cards of that suit. There are $4 \times \binom{13}{5}$ ways to make a flush. Then the probability is

$$\frac{4 \times \binom{13}{5}}{\binom{52}{5}} \approx 0.00198079231.$$

- #57
What is the probability that a poker hand contains exactly one pair (that is, a poker hand with exactly four different ranks)? Choose the four ranks, $\binom{13}{4}$ ways, and then choose one of these ranks to pair, 4 ways. For the pair, there are $\binom{4}{2}$ ways to choose the suits, and for the other three cards there are 4^3 ways to choose suits. Putting it all together, the probability of getting a pair is

$$\frac{\binom{13}{4} \times 4 \times \binom{4}{2} \times 4^3}{\binom{52}{5}} \approx 0.422569028$$

(Reasoning slightly differently, one might write $13 \times \binom{12}{3}$ rather than $\binom{13}{4} \times 4$.)

- #58
What is the probability that a poker hand contains cards of five different ranks but does not contain a flush or a straight?
Choose the five ranks, $\binom{13}{5}$ ways, then subtract the straights. There are 10 possible straights, A low to 10 low, so $\binom{13}{5} - 10$ ways to choose rank without a straight. Now, choose the four suits for each card, 4^5 ways, but subtract the flushes. There are 4 choices that lead to flushes: all clubs, diamonds, hearts, or spades. So there are $4^5 - 4$ ways to choose suits without having a flush. Putting it all together, the probability is

$$\frac{\left(\binom{13}{5} - 10\right)\left(4^5 - 4\right)}{\binom{52}{5}} \approx 0.50313971742.$$

- #63 Four (standard) dice (cubes with 1, 2, 3, 4, 5, 6, respectively, dots on their six faces), each of a different color, are tossed, each landing with one of its faces

up, thereby showing a number of dots. Determine the following probabilities:
(a) The probability that the total number of dots shown is 6
The number of solutions to $x_1 + x_2 + x_3 + x_4 = 6$ with $x_i \geq 1$ for $i = 1, 2, 3, 4$, is the same as the number of non-negative solutions to $y_1 + y_2 + y_3 + y_4 = 2$, or $\binom{5}{3}$. Note that all of these solutions actually correspond to possible values for the dice—since the total of x_i is 6, none of the dice can take a value more than 6. So the probability is

$$\frac{\binom{5}{3}}{6^4}.$$

(b) The probability that at most two of the dice show exactly one dot
The number of rolls where no dice show one dot: 5^4
The number of rolls where one dice shows one dot: $\binom{4}{1} \times 5^3$
The number of rolls where two dice show one dot: $\binom{4}{2} \times 5^2$
So the probability that at most two of the dice show exactly one dot is

$$\frac{5^4 + \binom{4}{1} \times 5^3 + \binom{4}{2} \times 5^2}{6^4}.$$

A perhaps simpler approach is to count the number of rolls where three dice or four dice show one dot, and subtract from total number of rolls. This leads to the equivalent expression

$$\frac{6^4 - 20 - 1}{6^4}.$$

(c) The probability that each die shows at least two dots

$$\frac{5^4}{6^4}$$

(d) The probability that the four numbers of dots shown are all different

$$\frac{6 \times 5 \times 4 \times 3}{6^4}$$

(e) The probability that there are exactly two different numbers of dots shown
There are two possibilities.

The first possibility is that there are three dice of one number, and one of another. There are $\binom{4}{3} \times 6 \times 5$ ways this can happen.

The second possibility is that there are two dice of one number, and two of another. There are $\binom{4}{2}$ ways to choose a pair of dice, 6 ways to assign a number to the pair and 5 ways to assign a number to its complement. But in this case, we have counted each possibility twice, since one could choose either the pair or its complement. So there are $\binom{4}{2}\binom{6}{2}$ ways this can happen.
Putting it together, we have that the probability that there are exactly two different numbers of dots shown is

$$\frac{\binom{4}{3} \times 6 \times 5 + \binom{4}{2}\binom{6}{2}}{6^4}.$$