Math 4575 : HW #2

Chapter 3: #4, 5, 9, 10, 11, 12, 18, 19, 20, 27

- #4
  Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, then there are always two which differ by 1.
  Consider the n pairs: $\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}$. If $n+1$ integers are chosen, then two must be chosen from the same pair, by the pigeonhole principle.

- #5
  Show that if $n+1$ distinct integers are chosen from the set $\{1, 2, \ldots, 3n\}$, then there are always two which differ by at most 2.
  Consider the n triples: $\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{2n-2, 2n-1, 2n\}$. If $n+1$ integers are chosen, then two must be from the same triple, by the pigeonhole principle. Any two integers in the same triple differ by at most 2.

- #9
  In a room there are 10 people, none of whom are older than 60 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that we can always find two groups of people (with no common person) the sum of whose ages is the same. Can 10 be replaced by a smaller number?
  There are $2^{10} - 1 = 1023$ nonempty subsets of the 10 people. The sum total of any nonempty group of people is at least 1 and at most 600. By the pigeonhole principle, two of these sums must be the same. Removing any common people from the two groups gives two (possibly smaller) groups with the same sum.
  Can 10 be replaced by a smaller number? Yes, 10 can be replaced by 9. We may assume without loss of generality that all the ages are different. Then the largest possible sum is $52 + 53 + \cdots + 60 = 504$. On the other hand, among 9 numbers, there are at $2^9 - 1 = 511$ nonempty subsets. Since $511 > 504$, again applying the pigeonhole principle, there are two subsets with the same sum. Removing any common people from the two groups gives two disjoint groups with the same sum.
  In fact, 9 can even be replaced by 8, but I don't know an easy proof of this! A student last year, Bill Varcho, did a computer search and found that there is no way to even have 8 people people ages 1 to 60, and have all the sums be distinct. On the other hand, his computer search also found that it is possible to have all the sums be distinct with 7 people. For example, if their ages are $1, 2, 4, 24, 40, 48, 56$, one may verify that the $2^7 - 1 = 127$ nonempty sums of ages are all distinct.

- #10
  A child watches TV at least one hour each day for seven weeks but, because of parental rules, never more than 11 hours in anyone week. Prove that there is some period of consecutive days in which the child watches exactly 20 hours of

TV. (It is assumed that the child watches TV for a whole number of hours each day.)
Let d_i denote the amount of TV the child watches on day i, for $1 \leq i \leq 49$. Let t_i denote the total amount of TV the child has watched, up through day i, i.e.

$$t_i = d_1 + d_2 + \cdots + d_i.$$

By the given assumptions, we have the chain of inequalities:

$$1 \leq t_1 < t_2 < \cdots < t_{49} \leq 77.$$

Adding 20 to everything, we also have

$$21 \leq t_1 + 20 < t_2 + 20 < \cdots < t_{49} + 20 \leq 97.$$

There are 98 numbers: $t_1, t_2, \ldots, t_{49}$ and $t_1 + 20, t_2 + 20, \ldots, t_{49} + 20$. Each of the numbers is an integer between 1 and 97, so two of the integers must be the same. They can not be from the set $t_1, t_2, \ldots, t_{49}$, since this is a strictly increasing sequence, and similarly, they can not be from the set $t_1 + 20, t_2 + 20, \ldots, t_{49} + 20$. So it must be one from each set. Then, if $t_i = t_j + 20$, necessarily $j < i$ and

$$d_{j+1} + \cdots + d_i = 20,$$

as desired.

- #11
  A student has 37 days to prepare for an examination. From past experience she knows that she will require no more than 60 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day, however), there is a succession of days during which she will have studied exactly 13 hours.
  This is similar to the previous problem. Let t_i denote the total number of hours of study, by the end of day i. Then

  $$1 \leq t_1 < t_2 < \cdots < t_{37} \leq 60,$$

  and

  $$14 \leq t_1 + 13 < t_2 + 13 < \cdots < t_{37} + 13 \leq 73.$$

  There are 74 numbers $t_1, t_2, \ldots, t_{37}$ and $t_1 + 13, t_2 + 13, \ldots, t_{37} + 13$, all between 1 and 73. We conclude as in #10.

- #12
  Show by example that the conclusion of the Chinese remainder theorem need not hold when m and n are not relatively prime.
  Many examples are possible. The point is to give an example of numbers a, b, m, and n, so that there does *not* exist any number x with $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$. Ideally you would also provide some reason why there does

not exist such an x. For example, there is no number x such that $x \equiv 0 \pmod 4$ and $x \equiv 3 \pmod 6$. Indeed, any such number x would have to be both even and odd, which is ludicrous.

- #18
  Prove that of any five points chosen within a square of side length 2, there are two whose distance apart is at most $\sqrt{2}$.
  Consider dividing the square up into four smaller identical squares, each of side length 1. The pigeonhole principle implies that for every set of five points, there must be at least one of the smaller squares containing at least two points. But the diameter of the small squares is $\sqrt{2}$, so any two points in such a square must be at distance at most $\sqrt{2}$.

- #19
  (a) Prove that of any five points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $1/2$.
  Divide the equilateral triangle into four equilateral triangles of side length $1/2$, and apply the pigeonhole principle.
  (b) Prove that of any ten points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $1/3$.
  Divide the equilateral triangle into nine equilateral triangles of side length $1/3$, and apply the pigeonhole principle.
  (c) Determine an integer m_n such that if m_n points are chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $1/n$.
  The above examples suggest that $m_n = n^2 + 1$ is such an integer. Indeed, one may divide the equilateral triangle of side length 1 into n^2 congruent equilateral triangles of side length $1/n$ with parallel lines. Then applying the pigeonhole principle gives the desired result.

- #20
  Prove that $R(3, 3, 3) \leq 17$.
  We are asked to show that in every 3-coloring the edges of the complete graph K_{17}, there is either a red K_3, a blue K_3, or a green K_3. Assume we are given a coloring. Let v be an arbitrary vertex in the graph. There are 16 edges meeting at v, so the pigeonhole principle implies that at least 6 of them are the same color. Assume without loss of generality (e.g. by permuting colors) that six red edges meet at v, connecting v to 6 vertices which we label $x_1, x_2, \ldots, x_6$.

  Now, if any pair of the x_i is connected by a red edge, we are done. The only other possibility is that every edge between them is either blue or green. But then since $R(3, 3) = 6$, there is either a blue triangle or a green triangle. In every case, there is either a red, blue, or green triangle.

  Comment: it happens that $R(3, 3, 3) \geq 17$ too, but this direction seems harder. Try it, for fun, if you want. Just to be clear: to show this, one must only exhibit an example of a 3-coloring of the edges of K_{16} with no monochromatic triangles.

- #27

  A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

  An equivalent formulation is: show that for any collection of $2^{n-1} + 1$ subsets of $\{1, 2, \ldots, n\}$, some pair of subsets has empty intersection.

  Let $\bar{A}$ denote the complement of the subset A in $\{1, 2, \ldots, n\}$. That is, an element is in $\bar{A}$ if and only if it is not in A. Then consider all the pairs $\{A, \bar{A}\}$. (These are really pairs, since the complement of a complement is the original set, i.e. $\bar{\bar{A}} = A$.)

  There are 2^{n-1} pairs of subsets. If you have a collection of $2^{n-1} + 1$ subsets, then by the pigeonhole principle there are two in the same pair. But a set and its complement do not intersect, so we are done.

  Comment: One can not hope to prove this with $2^{n-1} + 1$ replaced by any smaller number. Indeed, consider the collection of all subsets containing the element 1. There are 2^{n-1} subsets in the collection, and every pair has nonempty intersection.

  Comment: Several people misinterpreted this to mean that *all* the subsets have some element in common. That would be a completely different (and much easier) problem. Note that every pair having a common element does not imply that the whole collection has a common element. For example, consider

  $$\{1, 2\}, \{2, 3\}, \{3, 1\}.$$