Name: ____

Math 4575 : midterm #1
Spring 2018

No calculators or notes are allowed.

Please keep your phones off or in airplane mode until after the exam.

Before you start, please check to make sure that you have five problems.

You have 55 minutes to complete the exam, 1:50–2:45pm.

Relax. Have fun.

1. (a) How many 7-digit numbers have every digit either a 1, 2, or 3?

$$3^7$$

(b) How many of the numbers from part (a) never repeat the same digit twice consecutively?

$$3 \times 2^6$$

(c) How many of the numbers from part (a) have at least one of every digit 1, 2, and 3?

Let S be the set of all numbers in part (a), and for $i = 1, 2, 3$ let A_i denote the subset of sequences which do not have any i's.

We have $|S| = 3^7$, $|A_1| = |A_2| = |A_3| = 2^7$ for every i, $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_1 \cap A_3| = 1$, and $|A_1 \cap A_2 \cap A_3| = 0$. So by inclusion-exclusion, there are

$$3^7 - 3 \times 2^7 + 3$$

such sequences.

2. (a) Consider a three-dimensional cube whose dimensions are $7 \times 7 \times 7$. You are at the front-lower-left corner of the cube and wish to get to the back-upper-right corner 21 "blocks" away. How many different routes are there in which you walk exactly 21 blocks, moving forward, up, or right at each step?

$$\binom{21}{7,7,7}$$

(b) Simplify the sum

$$\sum_{k=1}^{n} k\binom{n}{k} = 1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}$$

to a closed-form formula.

One idea is to use the binomial theorem. First, expand $(1+x)^n$, then differentiate both sides with respect to x, and finally, set $x = 1$.

Another approach: this counts the number ways to choose a committee with any number of members, and to elect a chair of the committee. Counting another way: choose the chair first, n ways, and then for everyone else decide whether they are on the committee or not, 2^{n-1} ways.

With either method, the final answer is

$$n2^{n-1}.$$

3. (a) How many permutations are there of the letters

$$XXXYYYZZZZ?$$

(There are three X's, three Y's, and four Z's.)

This is given by the multinomial coefficient

$$\binom{10}{3,3,4}.$$

(b) How many permutations are there, if no two X's can be consecutive?

Permute the Y's and Z's $\binom{7}{3}$ ways, and then insert the X's into the 8 possible spaces including the ends $\binom{8}{3}$ ways. So in total there are

$$\binom{7}{3}\binom{8}{3}$$

ways.

(c) How many, if all three of the Y's have to be consecutive but no two X's can be consecutive?

Similar to the above, but glue all the Y's together. Now there are

$$\binom{5}{1}\binom{6}{3}$$

ways.

4. Four standard dice (cubes with $1, 2, \ldots, 6$ dots on their faces), each of a different color, are tossed. Each dice lands with one of its faces up, thereby showing a number of dots. Determine the following probabilities.

   (a) The probability that the four dice all show different numbers of dots.

   $$\frac{6 \times 5 \times 4 \times 3}{6^4}.$$

   (b) The probability that all of the dice show 5 or more dots.

   Every die shows 5 or 6 dots. Two possibilities for each die, so

   $$\frac{2^4}{6^4}.$$

   (c) The probability that the total number of dots is 9.

   Substituting $y_i = x_i - 1$, the number of solutions to $x_1 + x_2 + x_3 + x_4 = 9$ with $1 \leq x_i \leq 6$ is the same as the number of solutions to $y_1 + y_2 + y_3 + y_4 = 5$ with $0 \leq y_i \leq 5$. Stars and bars gives that there are $\binom{8}{3}$ solutions to the equation with non-negative integers $y_i \geq 0$. But we are luck and these are all also solutions with $0 \leq y_i \leq 5$. None of the integers y_i can be greater than 5, since otherwise the sum would be more than 5.

   So the final probability is

   $$\frac{\binom{8}{3}}{6^4}.$$

5. Prove that no matter how you 3-color the edges of the complete graph K_{17} with red, green, and blue, there is always a monochromatic triangle—i.e. there is either a red triangle, a blue triangle, or a green triangle.

   Let v be an arbitrary vertex. There are 16 edges meeting at v, so by the pigeonhole principle, at least 6 edges meeting there are the same color. Without loss of generality, suppose that vertices $v_1, v_2, \ldots, v_6$ are all connected to v by red edges.

   If any pair of $v_1, \ldots, v_6$ is connected by a red edge, we already have a red triangle and we're done. Otherwise all of these edges are green or blue. But this gives a 2-coloring of the edges of K_6. Since $R(3,3) = 6$, in this case there is either a green or blue triangle. In every case, there is a monochromatic triangle.

(Extra credit.) Show that $R(5, 5) \leq 70$.

First, recall that $R(s, 2) = 2$ for every $s \geq 2$ and $R(2, t) = 2$ for every $t \geq 2$.

5	5			
4	4			
3	3			
2	2	3	4	5
	2	3	4	5

Then repeatedly use the fact that $R(s, t) \leq R(s-1, t) + R(s, t-1)$ to fill in the grid. At each step, the Ramsey number in an empty square is at most the sum of the number below it and the number to its left.

5	5			
4	4			
3	3	6		
2	2	3	4	5
	2	3	4	5

This can all be done on one grid, of course. I am just illustrating it with multiple grids to show the steps. (Continued on the next page.)

5	5			
4	4	10		
3	3	6	10	
2	2	3	4	5
	2	3	4	5

5	5	15	35	
4	4	10	20	35
3	3	6	10	15
2	2	3	4	5
	2	3	4	5

5	5	15	35	70
4	4	10	20	35
3	3	6	10	15
2	2	3	4	5
	2	3	4	5