Math 4575 : HW #6

Chapter 7: #1, 2, 8, 31, 32, 33, 37, 38, 39, 40

- #1

  Let $f_0, f_1, f_2, \ldots$ denote the Fibonacci sequence. By evaluating each of the following expressions for small values of n, conjecture a general formula and then prove it, using mathematical induction and the Fibonacci recurrence. (Comment: we observe the convention that $f_0 = 0$, $f_1 = 1$, etc.)

  (a)
  $$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$$

  The proof is by induction. The formula holds for $n = 1$, and suppose it holds for some $n \geq 1$, adding f_{2n+1} to both sides gives

  $$f_1 + f_3 + \cdots + f_{2n+1} = f_{2n} + f_{2n+1} = f_{2n+2},$$

  so the identity holds for $n + 1$ as well. By induction, the result holds for all $n \geq 1$.

  (b)
  $$f_0 + f_2 + \cdots + f_{2n} = f_{2n+1} - 1$$

  The formula holds for $n = 0$, and suppose it holds for some $n \geq 0$, adding f_{2n+2} to both sides gives

  $$f_0 + f_2 + \cdots + f_{2n} + f_{2n+2} = f_{2n+1} + f_{2n+2} - 1 = f_{2n+3} - 1,$$

  so the identity holds for all $n \geq 0$ by induction.

  (c)
  $$f_0 - f_1 + f_2 - \cdots + (-1)^n f_n = (-1)^n f_{n-1} - 1$$

  Again, we proceed by induction. If we make the (reasonable) convention that $f_{-1} = 1$, then it holds for $n = 0$. One could also have the base case be $n = 1$. Assuming that the formula holds for n, add $(-1)^{n+1} f_{n+1}$ to both sides to obtain

  $$\begin{aligned} f_0 - f_1 + f_2 - \cdots + (-1)^{n+1} f_{n+1} &= (-1)^n f_{n-1} + (-1)^{n+1} f_{n+1} - 1 \\ &= (-1)^{n+1} (f_{n+1} - f_{n-1}) - 1 \\ &= (-1)^{n+1} f_n - 1, \end{aligned}$$

  as desired.

(d)

$$f_0^2 + f_1^2 + \cdots + f_n^2 = f_n f_{n+1}$$

The formula holds for $n = 0$. Now suppose it holds for n and add f_{n+1}^2 to both sides. We have

$$\begin{aligned} f_0^2 + f_1^2 + \cdots + f_{n+1}^2 &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}(f_n + f_{n+1}) \\ &= f_{n+1} f_{n+2}, \end{aligned}$$

so it holds with $n+1$ as well. By induction it holds for every $n \geq 0$.

- #2
Prove that the nth Fibonacci number is the integer that is closest to the number

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

We have the formula

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

so the desired result follows if we know that

$$\left|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right| < \frac{1}{2},$$

but this is clear. Indeed, $(1/\sqrt{5}) < 1/2$ and $\left|\left((1-\sqrt{5})/2\right)^n\right| \leq 1$ for $n \geq 0$.

- # 8
Consider a 1-by-n chessboard. Suppose we color each square of the chessboard with one of the two colors red and blue. Let h_n be the number of colorings in which no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then derive a formula for h_n.

We claim that $h_n = h_{n-1} + h_{n-2}$, $(n \geq 2)$, and $h_0 = 1$ and $h_1 = 2$.

The recurrence is checked as follows. Take a sequence of length n where $n \geq 2$. If the last square is blue, remove it to obtain a sequence of length $n-1$. If the last square is red, then the previous square must be blue, so remove both tiles to obtain a sequence of length $n-2$. This process is reversible: given a sequence of length $n-2$, one can append blue and red tile (in this order), and given a sequence of length $n-1$, one can append a blue tile. So we have $h_n = h_{n-1} + h_{n-2}$. The initial conditions are straightforward: there is one (empty) tiling of the 1-by-0 chessboard, and both tilings of the 1-by-1 chessboard are allowable, so $h = 1$ and $h_2 = 2$.

This has the same recurrence as the Fibonacci numbers, and almost the same initial values, except shifted. In other words $h_n = f_{n+2}$.

Then our formula for the Fibonacci numbers implies that

$$h_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}.$$

Other forms for the answer are possible, for example if you would rather have n in the exponent than $n+2$, after a little algebra we have

$$h_n = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^{n} + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^{n}.$$

- #31
  Solve the recurrence relation $h_n = 4h_{n-2}, (n \geq 2)$ with initial values $h_0 = 0$ and $h_1 = 1$.

  Writing the first few values, we have

  $$0, 1, 0, 4, 0, 16, 0, 64, 0, 250, 0 \ldots$$

  We can guess the formula, which then follows by strong induction.

  $$h_n = 0 \text{ if } n \text{ is even, and}$$

  $$h_n = 4^{(n-1)/2} = 2^{n-1} \text{ if } n \text{ is odd.}$$

  Other forms of the answer are possible. For example,

  $$h_n = \frac{1}{4}2^n - \frac{1}{4}(-2)^n.$$

  You can obtain this by solving the recurrence with standard methods, since it is linear homogeneous with constant coefficients.

- #32
  Solve the recurrence relation $h_n = (n+2)h_{n-1}, (n \geq 1)$ with initial value $h_0 = 2$.

  The first few terms of the sequence are

  $$2,\ 2\times 3,\ 2\times 3\times 4,\ 2\times 3\times 4\times 5, \ldots.$$

  This looks like the factorial sequence, but shifted. Indeed, we guess $h_n = (n+2)!$ and once we observe it, it follows immediately by strong induction.

- #33

  Solve the recurrence relation $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $(n \geq 3)$ with initial values $h_0 = 0,\ h_1 = 1$, and $h_2 = 2$.

  The characteristic polynomial is $x^3 - x^2 - 9x + 9$, which factors as

  $$(x+3)(x-3)(x-1)$$

  which has roots $\pm 3, 1$. A general solution to the recurrence is given by

  $$h_n = c_1 3^n + c_2(-3)^n + c_3.$$

  Evaluating at $n = 0, 1, 2$ gives

  $$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 1 &= 3c_1 - 3c_2 + c_3 \\ 2 &= 9c_1 + 9c_2 + c_3 \end{aligned}$$

  Straightforward linear algebra gives $c_1 = 1/3$, $c_2 = -1/12$, and $c_3 = -1/4$. So

  $$h_n = \frac{1}{3}3^n - \frac{1}{12}(-3)^n - \frac{1}{4}.$$

- #37

  Determine a recurrence relation for the number a_n of ternary strings (made up of 0's, 1's, and 2's) of length n that do not contain two consecutive 0's or two consecutive 1's. Then find a formula for a_n.

  The key observation is that a_n satisfies a recurrence, namely

  $$a_n = 2a_{n-1} + a_{n-2},\ (n \geq 2),$$

  with initial conditions $a_0 = 1$ and $a_1 = 3$. The justification for the recurrence is the following. Given a sequence of length $n-2$, one can append 22 to the end to obtain an allowable sequence of length n. Give a sequence of length $n-1$, one can append any number different from the last number in the current sequence to obtain an allowable sequence of length n. (If the sequence of length $n-1$ ends in 0, you can add a 1 or 2, etc.) It is easy to verify that every sequence of length n is obtain this way exactly once. Indeed, to reverse the algorithm, consider the last two digits. If they are double, they must be 22 and it came from a sequence of length $n-2$. If the digits are not doubled, then it came from a sequence of length $n-1$.

  Once we have this recurrence and initial values, we can use our standard techniques. In this case, after a little calculation we find

  $$a_n = \frac{1}{2}\left(1+\sqrt{2}\right)^{n+1} + \frac{1}{2}\left(1-\sqrt{2}\right)^{n+1}.$$

Other forms for the answer are possible. For example,

$$a_n = \left(\frac{1+\sqrt{2}}{2}\right)\left(1+\sqrt{2}\right)^n + \left(\frac{1-\sqrt{2}}{2}\right)\left(1-\sqrt{2}\right)^n.$$

- #38
Solve the following recurrence relations by examining the first few values for a formula and then proving your conjectured formula by induction.

(a) $h_n = 3h_{n-1}, (n \geq 1); h_0 = 1.$

$$1, 3, 9, 27, 81, 241, 729, \ldots$$

We guess $h_n = 3^n$. Once we make the guess, the proof holds by induction.

(b) $h_n = h_{n-1} - n + 3, (n \geq 1); h_0 = 2.$

$$2, 4, 5, 5, 4, 2, -1, -5, \ldots$$

We guess that h_n is quadratic, i.e. $h_n = an^2 + bn + c$ for some constants a, b, and c that don't depend on n. (We didn't talk about it in class, but the following idea is covered in this chapter of the book: the "difference sequence" for a polynomial of degree n should be a polynomial of degree $n-1$. If so, a little computation gives that $a = -1/2$, $b = 5/2$, and $c = 2$. Verifying the recurrence is now straightforward proof by induction. We have

$$h_n = -\frac{1}{2}n^2 + \frac{5}{2}n + 2.$$

(c) $h_n = -h_{n-1} + 1, (n \geq 1); h_0 = 0$

$$0, 1, 0, 1, 0, 1, 0, 1, \ldots$$

It seems that $h_n = 0$ for n even and $h_n = 1$ for n odd, easily proved by induction. (This answer may be written in various forms.)

(d) $h_n = -h_{n-1} + 2, (n \geq 1); h_0 = 1$

$$1, 1, 1, 1, 1, \ldots$$

It seems that $h_n = 1$ for every $n \geq 0$. The proof? By induction.

(e) $h_n = 2h_{n-1} + 1, (n \geq 1); h_0 = 1$

$$1, 3, 7, 15, 31, 63, \ldots$$

We try $h_n = 2^{n+1} - 1$. The result is easily verified by induction.

- #39
  Let h_n denote the number of ways to perfectly cover a $1 \times n$ board with monominoes and dominoes in such a way that no two dominoes are consecutive. Find, but do not solve, a recurrence relation and initial conditions satisfied by h_n.

  We claim that

  $$h_n = h_{n-1} + h_{n-3}, \ (n \geq 3)$$

  with initial conditions $h_0 = 1$, $h_1 = 1$, and $h_2 = 2$. Indeed, if the rightmost tile is a monomino, remove it to obtain a tiling of a $1 \times (n-1)$ board. If it is a domino and $n \geq 3$ then the domino must have been preceded by a monomino. So remove both the domino and monomino to obtain a tiling of an $1 \times (n-3)$ board.

  This algorithm is reversible. Given a tiling of a $1 \times (n-1)$ board, one can always add a monomino on the right. And given a tiling of a $1 \times (n-3)$ board, one can always add a monomino and then a domino. So this gives a bijection between sequences of length n and the disjoint union of sequences of length $n-1$ and $n-3$.

- #40
  Let a_n equal the number of ternary strings of length n made up of 0's, 1's, and 2's, such that the substrings 00, 01, 10, and 11 never occur. Prove that $a_n = a_{n-1} + 2a_{n-2}$, $(n \geq 2)$, with $a_o = 1$ and $a_1 = 3$. Find a formula for a_n.

  Given any allowable sequence of length $n-1$, one can append a 2 to the end to obtain an allowable sequence of length n. Given an allowable sequence of length $n-2$, one can append 21 or 20 an obtain an allowable sequence of length n.

  This algorithm is reversible. Given an allowable sequence of length $n \geq 2$, if it ends in 0 or 1, it must end with 20 or 21 respectively. Removing this gives a sequence of length $n-2$. If it ends in 2, one can remove the 2 to have a sequence of length $n-1$.

  Now, to solve the recurrence is straightforward. The characteristic polynomial is $x^2 - x - 2 = 0$, which has roots $x = 2$ and $x = -1$. The general form of the solution is $a_n = c_1 2^n + c_2(-1)^n$. Putting in the initial conditions allows us to solve for c_1 and c_2. We have

  $$a_n = \frac{4}{3}2^n - \frac{1}{3}(-1)^n.$$