Math 4575 : HW #8

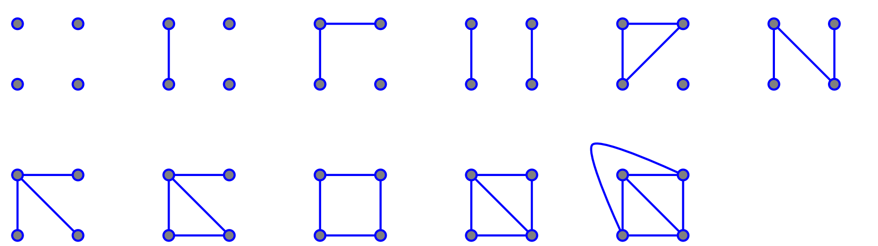
Chapter 11: #1, 2, 3, 4, 8, 9

- #1
  How many nonisomorphic graphs of order 1 are there? of order 2? of order 3? Explain why the answer to each of the preceding questions is ∞ for general graphs.

  There is one nonisomorphic graph of order 1, the edge empty graph. There are two graphs of order 2—either this is an edge, or there isn't. There are four graphs of order 3—in this case, again the isomorphism type is determined by the number of edges, which is either 0, 1, 2, or 3.

  The answer is ∞ in the case of general graphs (i.e. multigraphs), since there is no bound on the number of edges, even for a graph of order 1. A single vertex may have arbitrarily many loops.

- #2
  Determine each of the 11 nonisomorphic graphs of order 4, and give a planar representation of each.



- #3
  Does there exist a graph of order 5 whose degree sequence equals $(4, 4, 3, 2, 2)$?

  No, since the number of vertices of odd degree is always even. There is not even a multigraph with this degree sequence.

- #4
  Does there exist a graph of order 5 whose degree sequence equals $(4, 4, 4, 2, 2)$? a multigraph?

  There is not a graph of order 5 with this degree sequence. Indeed, if vertices 1, 2, and 3 each have degree 4 then they are each connected to every other vertex, so vertices 4 and 5 must each have degree at least 3.

  On the other hand, there is a multigraph with this degree sequence. Consider the graph that is the union of a triangle 123 and an edge 45. Then the degree sequence is $(2, 2, 2, 1, 1)$, so replacing each edge by a "double edge" results in degree sequence $(4, 4, 4, 2, 2)$.

- #8

  Let G be a graph with degree sequence $(d_1, d_2, \ldots, d_n)$. Prove that, for each k with $0 < k < n$,

  $$(*) \qquad \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

  We interpret the left side as counting incident pairs (v, e), here v is a vertex in the first k vertices of G, and e is an edge containing v.

  Now we try to count incident pairs (v, e), but by summing over edges rather than over vertices. Every edge counted in an incident pair is in one of two categories: either both its ends are in the first k vertices, or only one end. There are at most $\binom{k}{2}$ edges with both ends on first k vertices, and each such edge would get counted twice, once for each end of the edge, so these edges contribute at most $k(k-1)$ to the sum.

  Every other edge has one end on one of the last $n-k$ vertices, and the other end on one of the first k vertices. For $i = k+1, \ldots, n$, the contribution of vertex i to the count of incident pairs can be at most k, since each edge must have its other end in the first k vertices. On the other hand we also note that its contribution can be at most d_i, since that is its total degree. Since both these bounds hold, we have that the contribution from vertex i is at most $\min\{k, d_i\}$.

  Putting it all together, we have the desired inequality.

  Comment: the Erdős–Gallai theorem is a famous theorem in graph theory, that a necessary and sufficient condition for $d_1 \geq d_2 \geq \ldots d_n \geq 0$ to be the degree sequence of graph is that (1) the sum $d_1 + d_2 + \cdots + d_n$ be even, and (2) the inequality $(*)$ be satisfied for every k. We have shown in this exercise that the condition is necessary, but showing that it is sufficient is harder.

- #9

  Draw a connected graph whose degree sequence equals $(5, 4, 3, 3, 3, 3, 3, 2, 2)$.