Review for midterm #2

1 Enumerative combinatorics

1.1 Some topics we've covered

- 1. Inclusion-exclusion
- 2. Solving a linear, homogeneous recursion relation.
- 3. Generating functions
- 4. Exponential generating functions
- 5. Fibonacci numbers.
- 6. Catalan numbers.
- 7. Stirling numbers of the second kind

 $\binom{n}{k}$

8. Partition numbers

$$p_k(n)$$
 and $p(n)$.

Exercise 1 Compute a closed form for the the generating function for each of the following series.

1. 1. $1 + x + x^{2} + x^{3} + \dots$ 2. $x + 2x^{2} + 3x^{3} + 4x^{4} + \dots$ 3. $x + 4x^{2} + 9x^{3} + 16x^{4} + \dots$ 4. $x + x^{2} + 2x^{3} + 3x^{4} + 5x^{5} + \dots$

(Fibonacci numbers)

5.

$$1 + x + 2x^2 + 5x^3 + 14x^4 + \dots$$

(Catalan numbers)

Exercise 2 Derive a formula for the nth Fibonacci number. Use any method you like.Exercise 3 Derive a formula for the nth Catalan number. Use any method you like.

1.2 The twelvefold way

A lot of enumerative combinatorics ideas are captured by the "Twelvefold Way." We count the number of ways to out $n \ge 1$ balls into $k \ge 1$ boxes, under various assumptions. The main variants are:

- 1. The balls are distinguishable or not. (2 possibilities.)
- 2. The boxes are distinguishable or not. (2 possibilities.)
- 3. Every box has to get at least one ball, at most one ball per box, or there are no restrictions. (3 possibilities.)

The name "Twelvefold Way" comes from these $2 \times 2 \times 3 = 12$ possibilities.

A slightly more precise way of thinking about it is as follows. We count either the number of functions $f : [n] \rightarrow [k]$ under various restrictions, or the *equivalence classes* of functions. For example, we say the "balls are indistinguishable" if we count functions modulo the following equivalence relation.

We consider functions $f_1, f_2 : [n] \to [k]$ equivalent if there exists a permutation (i.e. a bijection) $\sigma : [n] \to [n]$. such that

$$f_1 = f_2 \circ \sigma.$$

Exercise 4 Check that this is an equivalence relation! That is, it is reflexive, symmetric, and transitive.

Similarly, we say the "boxes are indistinguishable" if we consider functions f_1, f_2 : $[n] \rightarrow [k]$ equivalent if there exists a permutation $\tau : [k] \rightarrow [k]$. such that

$$f_1 = \tau \circ f_2.$$

Note the difference in notation between pre-composing with a permutation and post-composing!

We say that the "balls *and* boxes are indistinguishable" if we consider functions $f_1, f_2 : [n] \to [k]$ equivalent if there exists a permutation $\sigma : [n] \to [n]$ and a permutation $\tau : [k] \to [k]$ such that

$$f_1 = \tau \circ f_2 \circ \sigma.$$

Exercise 5 *Check that this is an equivalence relation!*

Finally saying "every box has to get at least one ball" corresponds to restricting to counting surjective functions, and saying "at most one ball per box" corresponds to counting injective functions.

Conceptually, the thing that makes the biggest difference is whether we are counting functions f or equivalence classes of functions and what kinds of equivalences we have to deal with. So we organize our thoughts this way. Let's warm up with counting functions without worrying about equivalences. That is, we count the number of ways to put balls into boxes, if both boxes and balls are distinguishable.

1.2.1 Boxes and balls are distinguishable

1. no restriction on f

 k^n

2. *f* is injective

$$k(k-1)\dots(k-n+1)$$

3. *f* is surjective

$$\binom{n}{k}k!$$

First, partition the set [n] into k nonempty subsets. The number of ways to do this is counted by Stirling numbers of the second kind. Now send all the elements in a subset to some number 1, 2, ..., k. We need a bijection between the subsets in the set partition, since f is surjective. So there are k! ways to do this.

1.2.2 Balls are indistinguishable, boxes are distinguishable

1. no restriction on f

$$\binom{n+k-1}{k-1}$$

Think of the the n indistinguishable balls as pieces of candy. The k boxes are the (distinguishable!) flavors of candy. So this is just "stars and bars" counting!

2. *f* is injective

 $\binom{k}{n}$

Injective is a big restriction here, which totally changes the flavor of the problem. It's like getting candy, but you are only allowed to get at most one piece of each type. For balls and boxes, just choose which n boxes get a ball. These boxes can only get one ball, since f is injective. Note in particular that this quantity is zero if n > k.

3. *f* is surjective

$$\binom{n-1}{k-1}$$

It's stars and bars counting again, but now you are counting non-negative integer solutions to $x_1 + \cdots + x_k = n$ with the additional constraint that $x_i \ge 1$ for $i = 1, 2, \ldots, k$. So after a change of variables $y_i = x_i - 1$, we are back to usual stars and bars.

1.2.3 Balls are distinguishable, boxes are indistinguishable

The way to think about this case is in counting terms of partitions of a set [n] into subsets. Here, it might be best to look at the surjective case first, and then compare the other two cases.

1. no restriction on f

$$\sum_{i=1}^k \binom{n}{i}$$

Partition [n] into at most k nonempty subsets. Here the index of summation i is ranging through all possibilities for the number of subsets.

2. *f* is injective

 $[n \leq k]$

Here we use the Iverson bracket, whose value is 1 if the expression inside is true, and 0 otherwise. If $n \le k$ then one has to partition [n] into subsets of size 1 and there is only one way to do this. If n > k there are no injective maps.

3. (*) *f* is surjective

 $\binom{n}{k}$

The number of ways to parition [n] into k nonempty subsets.

1.2.4 Balls are indistinguishable, boxes are indistinguishable

Think about this case as partitions of a *number* n, rather than partitions of a set [n]. Recall that the partition of a number n is a solution to

$$x_1 + x_2 + \dots + x_k = n$$

for some $x_1 \ge x_2 \ge \cdots \ge x_k \ge 1$.

Here we use the notation $p_k(n)$ to denote the number of solutions. Again, it might be best to look at the surjective case first.

1. no restriction on f

$$\sum_{i=1}^{k} p_i(n)$$

Partition the number n into at most k nonzero summands. Here the index of summation i is ranging through all possibilities for the number of summands. Another expression for this case is $p_k(n + k)$ —recall, we showed in class that the number of partitions of n + k into exactly k parts is the same as the number of partitions of n into at most k parts.

2. *f* is injective

 $[n \le k]$

Here, again, we use the Iverson bracket whose value is 1 if the expression inside is true, and 0 otherwise. If $n \le k$ then one has to partition n into summands of order 1, and there is only one way to do this. If n > k there are no injective maps.

3. (*) *f* is surjective

 $p_k(n)$

Since balls are indistinguishable, all that matters is the number of balls in each box, and since boxes are also indistinguishable, we may reorder the boxes so that the number of balls in each box is in weakly decreasing order, forming a partition of n into k parts.

2 Graph theory

- 1. What does it mean for two graphs to be isomorphic?
- 2. State the characterization of which multi-graphs contain an Eulerian cycle. Similarly, which multi-graphs contain an Eulerian path?
- 3. Various conditions that guarantee Hamiltonian or non-Hamiltonian.
- 4. Basic facts about trees
 - (a) Every finite tree on $n \ge 2$ vertices has a pendant vertex, i.e. a degree-one vertex.
 - (b) In a tree, every pair of vertices is connected by a unique path.
 - (c) A tree on n vertices has n 1 edges
 - (d) A tree is a maximal cycle-free graph
 - (e) A tree is a minimal connected graph
- 5. Graph coloring
 - (a) What is the chromatic number of a cycle C_k ?
 - (b) Trees are bipartite (2-colorable).
 - (c) Clique number and independence number lower bounds on chromatic number.

Exercise 6 In a graph of with n vertices, every vertex v has degree

 $0 \le \deg(v) \le n - 1.$

Argue that a graph must have two vertices of the same degree.