Math 4575 : HW #10

Chapter 11: #53, 60, 64
Chapter 12: #4, 5, 6, 7, 10, 12, 13, 20

Chapter 11

- #53
  Prove that a graph is a tree if and only if it does not contain any cycles, but the insertion of any new edge always creates exactly one cycle.

  First suppose that G is a tree. Clearly it does not contain any cycles. We claim that the insertion of any new edge creates exactly one cycle. A fundamental fact about trees is that every pair of vertices u, v are connected by a unique path $P(u, v)$. If $\{u, v\}$ is an inserted edge, then taking the union of $\{u, v\}$ and $P(u, v)$ gives a cycle. Moreover, this cycle is unique, by the uniqueness of the path $P(u, v)$.

  Now suppose that G has no cycles but the insertion of any new edge always creates exactly one cycle. To show that G is a tree, we need only show that it is connected. But this is clear—if G had two or more components, then we could insert an edge between them and merge two components and since it is a bridge edge, this would not create any cycles.

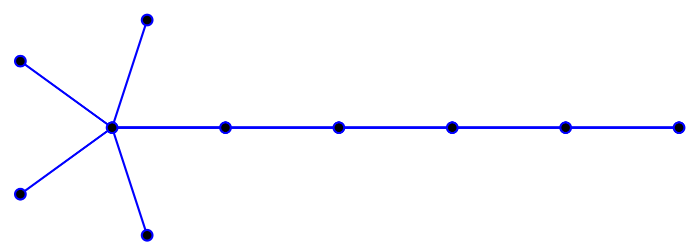
- #60
  Let G be a forest of k trees. What is the fewest number of edges that can be inserted in G in order to obtain a tree?

  When adding a single edge, the number of components either stays the same or drops by one, depending on whether the two endpoints of the edge are in the same connected component or different components. So the number of components can never decrease by more than one with adding an edge. Then at least $k - 1$ edges are necessary, in order to obtain a tree. On the other hand, $k - 1$ edges are also sufficient, since whenever a graph is disconnected, there is some edge that can be inserted to join two components.

- #64

  For each integer $n \geq 3$ and for each integer k with $2 \leq k \leq n-1$, construct a tree of order n with exactly k pendent vertices.

  Fix $n \geq 3$, and we will vary k from 2 to $n-1$. At the endpoints of the interval, it is easy to describe a tree with n vertices and k pendent vertices: if $k = 2$, then we have a path, and if $k = n-1$ then we have a star. Now we can interpolate between these endpoints. Consider $k-1$ paths of length 1 and 1 path of length $n-k$, all meeting at a single vertex. This is a tree of order n with k pendent vertices. The case $n = 10$, $k = 5$ is illustrated below.



Chapter 12

- #4

  Prove that the chromatic number of a cycle graph C_n of odd length equals 3.

  First, we note that 3 colors suffice. For example, we can color 1, 2, 1, 2, in cyclic order, until reaching the last vertex, and then coloring the last vertex 3. On the other hand 3 colors are also sufficient. A graph is 2-colorable if any only if it has no odd cycles.

- #5

  Determine the chromatic numbers of the following graphs.

  (Illustration in Brualdi. Left: 2, middle: 3, right: 4. For the last graph, we see that it is K_5 minus an edge. Clearly 4 colors are necessary and sufficient. (See for example, #10.)

- #6

  Prove that a graph with chromatic number equal to k has at least $\binom{k}{2}$ edges.

  Let $k = \chi(G)$ and consider a proper k-coloring of G. For every pair of colors $i, j \in \{1, 2, \ldots k\}$, there must exist some vertex x colored i and a vertex y colored j, such that x and y are adjacent. Otherwise, we could merge i and j into one color class and still have a proper coloring but with fewer colors, a contradiction to $\chi(G) = k$.

  So for every pair of colors, there is an edge between the color classes. This gives at least $\binom{k}{2}$ edges, as desired, since all these edges must be distinct.

- #7

  Prove that the greedy algorithm always produces a coloring of the vertices of $K_{m,n}$ in two colors ($m, n \geq 1$).

  The vertices of $K_{m,n}$ are $S \coprod T$, where $S = [m]$ and $T = [n]$, and the edges are $\{(s,t) | s \in S, t \in T\}$. Consider an arbitrary order on the vertices: $v_1, v_2, \ldots, v_{m+n}$. The first vertex v_1 gets colored with color 1. Without loss of generality assume $v_1 \in S$. We claim that from this point on, every vertex in S gets color 1 and every vertex in T gets color 2. We proceed by induction.

  The induction is that every vertex in S gets colored 1 and every vertex in T gets colored 2. Suppose it is true for vertices v_i with $i \leq N - 1$, then we show it it true for vertex v_N. When adding vertex v_N, if it is in S then by the induction hypothesis, all the vertices in T so far have gotten color 2—since V_N has all its neighbors in T, color 1 is still available. On the other hand if v_N is in T then it is connected to v_1 so it can not receive color 1. On the other hand, all other vertices in S have also gotten color 1 by the induction hypothesis, and v_N has no neighbors in T, so color 2 is available.

  Comment: show that the same does not hold for an arbitrary bipartite graph. Consider the 6 cycle. Can you give an ordering of the vertices so that the greedy algorithm uses more than 2 colors.

- #10

  What is the chromatic number of the graph obtained from K_n by removing one edge?

  Suppose edge $\{u, v\}$ is deleted. Then u and v can be given the same color, so $n - 1$ colors are sufficient for coloring. On the other hand $n - 1$ colors are still necessary, since the clique number is still at least $n - 1$. (Through out vertex v, then the remaining vertices span a clique, so they all require different colors.)

- #12

  What is the chromatic number of the graph obtained from K_n by removing two edges with a common vertex?

  Again, $n - 1$ colors are still necessary. If $\{u, v\}$ and $\{v, w\}$ are the two edges deleted, then all the vertices except v span a clique. On the other hand $n - 1$ colors are sufficient—color this clique, and then give v the same color as either u or w.

- #13

  What is the chromatic number of the graph obtained from K_n by removing two edges without a common vertex?

  Suppose that edges $\{u, v\}$ and $\{x, y\}$ are deleted. Then deleting u and x gives a clique of order $n - 2$, so at least $n - 2$ colors are required. On the other hand, one can color this clique with $n - 2$ colors and then give v the same color as u and y the same color as x. So $n - 2$ colors are also sufficient.

- #20
  Give an example of a planar graph with chromatic number 4 that does not contain a K_4 as an induced subgraph.