Math 4575 : HW #11

1. (a) Prove that the complement of a disconnected graph is connected.

   (b) Is it true that the complement of a connected graph is necessarily disconnected?

2. Show that a connected planar graph with $v \geq 3$ vertices must have at least 3 vertices of degree at most 5.

3. Give an example of a finite planar graph where every vertex has degree at least 5.

4. Suppose that P is a convex polyhedron in 3-dimensional space. Suppose that every face of P is either a pentagon or hexagon, and that exactly three faces meet at every vertex. How many faces are pentagons?

5. (a) Show that for a connected bipartite planar graph G with $v \geq 5$ vertices and e edges,
   $$e \leq 2v - 4.$$
   You may use the fact that, as long as $v \geq 5$, every face in a connected bipartite planar graph must contain at least 4 edges in its boundary.

   (b) Show that the complete bipartite graph $K_{3,3}$ is not planar.

6. (a) Show that if G contains a subgraph H, and H is not planar, then G is not planar either.

   (b) Determine for which n the complete graph K_n is planar.

   (c) Determine for which m, n the complete bipartite graph $K_{m,n}$ is planar.

7. Define the *hypercube graph* Q_d as follows. The vertices of Q_d correspond to binary sequences of length d. So there are 2^d vertices. Then the edges are defined by declaring that vertices x and y are adjacent, whenever the corresponding sequences differ in exactly one coordinate.

   For example, Q_3 has 8 vertices, corresponding to
   $$(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1).$$
   Vertex $(1,0,1)$ is adjacent to $(0,0,1)$, $(1,1,1)$, and $(1,0,0)$.

   (a) Find a formula for the number of edges in Q_d.

   (b) Show that Q_d is bipartite for every $d \geq 1$.

   (c) Show that Q_d is planar if and only if $d \leq 3$.

8. (a) Show that even if you delete two edges from the complete graph K_6, no matter which two edges you delete it is still not planar.

   (b) Show that it is possible to delete three edges from K_6 to obtain a planar graph.