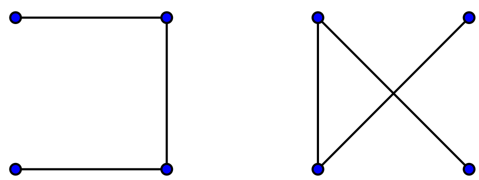
Math 4575 : HW #11

1. (a) Prove that the complement of a disconnected graph is connected.

   If $G = (V, E)$ is disconnected, then we can partition the vertices into two nonempty subsets $V = A \coprod B$ so that there are no edges between A and B. Then in the complement $\bar{G}$, there is a complete bipartite graph with parts A and B. Since $K_{m,n}$ is connected for every $m, n \geq 1$, we have that $\bar{G}$ is connected.

   (b) Is it true that the complement of a connected graph is necessarily disconnected?

   No. Consider the following complementary pair of connected graphs on four vertices.



2. Show that a connected planar graph with $v \geq 3$ vertices must have at least 3 vertices of degree at most 5.

   Let $d_1, d_2, \ldots, d_v$ be the vertex degrees, and assume without loss of generality that $1 \leq d_1 \leq d_2 \leq \cdots \leq d_v$. (We know that every vertex has degree ≥ 1, since the graph is connected.) By counting incident vertex-edge pairs two different ways, we have the well-known identity

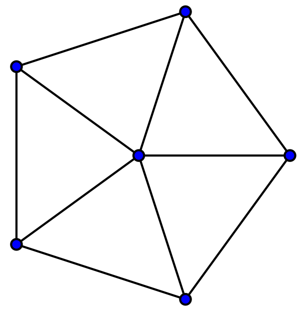
$$d_1 + d_2 + \cdots + d_v = 2e.$$

   We know that $e \leq 3v - 6$ since G is a planar graph and $v \geq 3$, so $2e \leq 6v - 12$. But $2e$ is the sum of the vertex degrees, so

$$d_1 + d_2 + \cdots + d_v \leq 6v - 12.$$

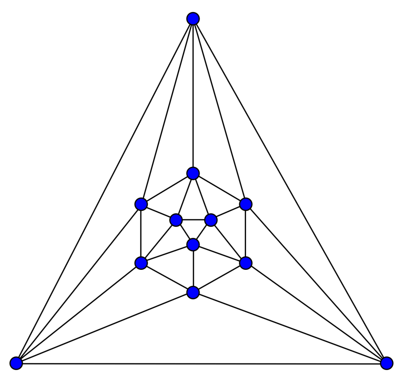
   Suppose by way of contradiction that $d_3 \geq 6$. Since $1 \leq d_1 \leq d_2$, the sum of all the vertex degrees is at least $6(v - 2) + 2 = 6v - 10$, contradiction to the inequality.

3. Give an example of a planar graph with chromatic number 4 that does not contain K_4 as a subgraph.



4. Give an example of a finite planar graph where every vertex has degree at least 5.

The icosahedron is the smallest such example. It is drawn as a planar graph below.



5. Suppose that P is a convex polyhedron in 3-dimensional space. Suppose that every face of P is either a pentagon or hexagon, and that exactly three faces meet at every vertex. How many faces are pentagons?

By stereographic projection, P corresponds to a planar graph. We have the Euler formula

$$v - e + f = 2. \tag{1}$$

Since every vertex has degree 3, we have

$$2e = 3v. \tag{2}$$

Finally, let f_5 denote the number of pentagon faces and f_6 the number of hexagons. Since every face is either a pentagon or a hexagon, we have that

$$f = f_5 + f_6. \tag{3}$$

By counting incident edge-face pairs two different ways, we also have that

$$2e = 5f_5 + 6f_6. \tag{4}$$

Multiplying (1) by 2 and adding to (2) gives

$$v = 2f - 4. \tag{5}$$

Combining (2) and (4) gives

$$3v = 5f_5 + 6f_6. \tag{6}$$

Multiplying (5) by 3 and then combining with (6) gives

$$6f - 12 = 5f_5 + 6f_6. \tag{7}$$

Finally, multiplying (3) by 6 and subtracting (7) gives

$$f_5 = 12,$$

so there are 12 pentagonal faces.

6. (a) Show that for a connected bipartite planar graph G with $v \geq 5$ vertices and e edges,

$$e \leq 2v - 4.$$

You may use the fact that, as long as $v \geq 5$, every face in a connected bipartite planar graph must contain at least 4 edges in its boundary.

Since G is connected and planar, we have the Euler formula

$$v - e + f = 2.$$

On the other hand, counting incident edge-face pairs in two different ways, we have that

$$2e \geq 4f.$$

Multiplying the first equation by 4, and adding to the inequality gives

$$4v - 2e \geq 8,$$

and dividing by 2 gives the desired result.

(b) Show that the complete bipartite graph $K_{3,3}$ is not planar.

We have that $v = 6$ and $e = 9$. Since $v \geq 5$ and these do not satisfy the inequality $e \leq 2v - 4$, the graph can not be planar.

7. (a) Show that if G contains a subgraph H, and H is not planar, then G is not planar either.

   This is clear. Any planar representation of G could just be "restricted" to a planar representation of an arbitrary subgraph H.

   (b) Determine for which n the complete graph K_n is planar.

   Since K_4 is planar, and K_5 is not, by part (a) we have that K_n is planar if and only if $n \leq 4$.

   (c) Determine for which m, n the complete bipartite graph $K_{m,n}$ is planar.

   Since $K_{3,3}$ is not planar, but $K_{2,n}$ is planar for every n, we have that $K_{m,n}$ is planar if and only if $m \leq 2$ or $n \leq 2$.

8. Define the *hypercube graph* Q_d as follows. The vertices of Q_d correspond to binary sequences of length d. So there are 2^d vertices. Then the edges are defined by declaring that vertices x and y are adjacent, whenever the corresponding sequences differ in exactly one coordinate.

   For example, Q_3 has 8 vertices, corresponding to

   $$(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1).$$

   Vertex $(1,0,1)$ is adjacent to $(0,0,1)$, $(1,1,1)$, and $(1,0,0)$.

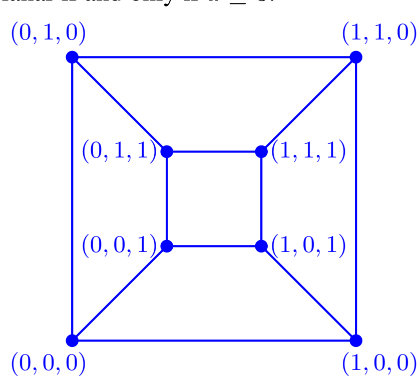
   (a) Find a formula for the number of edges in Q_d.

   Every vertex has degree d. Since the sum of the vertex degrees is twice the number of edges, there are $2^{d-1}d$ edges in Q_d.

   (b) Show that Q_d is bipartite for every $d \geq 1$.

   Let $A = \{\text{vertices with an odd number of ones}\}$ and $B = \{\text{vertices with an even number of ones}\}$. This is easily seen to be a bipartition—there are no edges between pairs of vertices in A or in B, since any such pair differs in at least two coordinates.

   (c) Show that Q_d is planar if and only if $d \leq 3$.



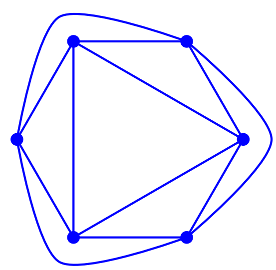
We know that Q_4 is bipartite by part (b), and by part (a) we have that $v = 16$ and $e = 32$. But for a bipartite planar graph with $v \geq 5$, we have $e \leq 2v - 4$ by problem #6. So Q_4 must not be planar.

9. (a) Show that even if you delete two edges from the complete graph K_6, no matter which two edges you delete it is still not planar.

   For K_6, we have $v = 6$ and $e = 15$. If you delete two edges, then $e = 13$. But this does not satisfy $e \leq 3v - 6$.

   (b) Show that it is possible to delete three edges from K_6 to obtain a planar graph.

   On the other hand, if you delete the appropriate 3 edges, the resulting graph becomes planar. Behold.



   Comment: this is the octahedron, realized as a planar graph.