# Math 6501 : Problem set 2 

due: Monday, October 27

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\begin{aligned}
(*) & =\text { required } \\
(* *) & =\text { optional } \\
(* * *) & =\text { unsolved }
\end{aligned}
$$

Please talk to each other and use any sources that you'd like, but please cite your sources.

1. (*) Let $V=\{0,1\}^{d}$, i.e. $V$ is the set of all $0-1$ sequences of length $d$. The graph on $V$ in which two such sequences are joined by an edge if and only if they differ in exactly one position is called the $d$-dimensional cube and is sometimes denoted $Q^{d}$.
(a) Determine the number of edges, average degree, diameter, and girth of $Q^{d}$.
(b) For which $d$ is $Q^{d}$ planar?
2. (*) Show that a connected, plane graph with $v \geq 3$ must have at least three vertices of degree $\leq 5$.
3. (*) Show that a finite tree with a vertex of degree $d$ must have at least $d$ vertices of degree one.
4. (*) Let $X$ be a graph with $\Delta(X) \leq 3$. Show that $G$ contains $X$ as a topological minor if and only if $G$ contains $X$ as a minor.
5. (*) Prove that for a bipartite, simple, planar graph with $v \geq 3$ vertices and $e$ edges,

$$
e \leq 2 v-4
$$

and conclude that the complete bipartite graph $K_{3,3}$ is not planar. If it is helpful, you may use the following fact: in a 2 -connected plane graph, the boundary of every face is a cycle.
6. (*) Suppose that $P$ is a convex 3 -dimensional polytope, such that every vertex has degree 3 and every face is either a pentagon or a hexagon. How many pentagonal faces does $P$ have?
7. (*) Suppose that $g=2 r+1$ is odd, and $\delta \geq 1$. Define

$$
n_{0}(\delta, g)=1+\delta \sum_{i=0}^{r-1}(\delta-1)^{i}
$$

Show that a graph of minimum degree $\delta$ and girth $g$ has at least $n_{0}(\delta, g)$ vertices.
8. (a) $(*)$ Classify finite trees with exactly 5 leaves, up to homeomorphism.
(b) $(* *)$ Classify trees with exactly 6 or 7 leaves, up to homeomorphism.
(c) $(* *)$ Show that the number of homeomorphism types of trees with $k$ leaves is at least $e^{C \sqrt{k}}$ for some constant $C>0$.
9. (*) Let $\operatorname{cr}(G)$ denote the crossing number of $G$. Prove the following elementary bounds on the crossing number of the complete graph $K_{n}$ :

$$
\frac{1}{5}\binom{n}{4} \leq \operatorname{cr}\left(K_{n}\right) \leq\binom{ n}{4}
$$

for $n \geq 5$. Note that the lower bound is better than what is obtained from the Crossing Number Inequality. (Hint: for the lower bound, use the fact that $K_{5}$ is not planar.)
10. $(* *)$ A graph is said to be outer-planar if it can be embedded in the plane so that every vertex is in the boundary of the outer (unbounded) face. Show that $G$ is outer-planar if and only if $G$ contains no $K_{4}$ or $K_{2,3}$ minors.
11. $(* *)$ Kruskal's tree theorem tells us that the class of finite trees, partially ordered by the relation of "topological minor" is a well quasi-order. There is no infinite anti-chain. Can you construct (or at least prove the existence) of arbitrarily large finite anti-chains? That is, for every $N$, describe a set of $N$ finite trees $T_{1}, T_{2}, \ldots, T_{n}$, such that none is a topological minor of any other.
12. $(* *)$ Give an example of an infinite sequence of graphs $\left\{G_{i}\right\}$ so that no graph in the sequence is a topological minor of any other graph in the sequence.
13. $(* * *)$ Fáry's Theorem is that every planar graph has a straight-line representation. Does every planar graph have a straight-line representation in which all edge lengths are integers?
14. (a) $(* * *)$ Let $X$ be a countably infinite graph. Show that $X$ contains itself as a proper minor.
(b) $(* *)$ Show that the above would imply the minor theorem of Robertson and Seymour: that given any infinite sequence of finite graphs $G_{1}, G_{2}, \ldots$, there exist indices $i<j$ such that $G_{i}$ is a minor of $G_{j}$.

