## Math 6501 : Problem set 2

due: Monday, October 27

(\*) = required (\*\*) = optional (\*\*\*) = unsolved

Please talk to each other and use any sources that you'd like, but please cite your sources.

- 1. (\*) Let  $V = \{0, 1\}^d$ , i.e. V is the set of all 0 1 sequences of length d. The graph on V in which two such sequences are joined by an edge if and only if they differ in exactly one position is called the *d*-dimensional cube and is sometimes denoted  $Q^d$ .
  - (a) Determine the number of edges, average degree, diameter, and girth of  $Q^d$ .
  - (b) For which d is  $Q^d$  planar?
- 2. (\*) Show that a connected, plane graph with  $v \ge 3$  must have at least three vertices of degree  $\le 5$ .
- 3. (\*) Show that a finite tree with a vertex of degree d must have at least d vertices of degree one.
- 4. (\*) Let X be a graph with  $\Delta(X) \leq 3$ . Show that G contains X as a topological minor if and only if G contains X as a minor.
- 5. (\*) Prove that for a bipartite, simple, planar graph with  $v \ge 3$  vertices and e edges,

 $e \le 2v - 4,$ 

and conclude that the complete bipartite graph  $K_{3,3}$  is not planar. If it is helpful, you may use the following fact: in a 2-connected plane graph, the boundary of every face is a cycle.

- 6. (\*) Suppose that *P* is a convex 3-dimensional polytope, such that every vertex has degree 3 and every face is either a pentagon or a hexagon. How many pentagonal faces does *P* have?
- 7. (\*) Suppose that g = 2r + 1 is odd, and  $\delta \ge 1$ . Define

$$n_0(\delta, g) = 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i$$

Show that a graph of minimum degree  $\delta$  and girth g has at least  $n_0(\delta, g)$  vertices.

- 8. (a) (\*) Classify finite trees with exactly 5 leaves, up to homeomorphism.
  - (b) (\*\*) Classify trees with exactly 6 or 7 leaves, up to homeomorphism.
  - (c) (\*\*) Show that the number of homeomorphism types of trees with k leaves is at least  $e^{C\sqrt{k}}$  for some constant C > 0.
- 9. (\*) Let cr(G) denote the crossing number of G. Prove the following elementary bounds on the crossing number of the complete graph  $K_n$ :

$$\frac{1}{5} \binom{n}{4} \le cr(K_n) \le \binom{n}{4}$$

for  $n \ge 5$ . Note that the lower bound is better than what is obtained from the Crossing Number Inequality. (Hint: for the lower bound, use the fact that  $K_5$  is not planar.)

- 10. (\*\*) A graph is said to be *outer-planar* if it can be embedded in the plane so that every vertex is in the boundary of the outer (unbounded) face. Show that G is outer-planar if and only if G contains no  $K_4$  or  $K_{2,3}$  minors.
- 11. (\*\*) Kruskal's tree theorem tells us that the class of finite trees, partially ordered by the relation of "topological minor" is a well quasi-order. There is no infinite anti-chain. Can you construct (or at least prove the existence) of arbitrarily large finite anti-chains? That is, for every N, describe a set of N finite trees  $T_1, T_2, \ldots, T_n$ , such that none is a topological minor of any other.
- 12. (\*\*) Give an example of an infinite sequence of graphs  $\{G_i\}$  so that no graph in the sequence is a topological minor of any other graph in the sequence.
- 13. (\* \* \*) Fáry's Theorem is that every planar graph has a straight-line representation. Does every planar graph have a straight-line representation in which all edge lengths are integers?
- 14. (a) (\* \* \*) Let X be a countably infinite graph. Show that X contains itself as a proper minor.
  - (b) (\*\*) Show that the above would imply the minor theorem of Robertson and Seymour: that given any infinite sequence of finite graphs  $G_1, G_2, \ldots$ , there exist indices i < j such that  $G_i$  is a minor of  $G_j$ .