Random simplicial complexes

MSRI Introductory Workshop: Geometric and Topological Combinatorics
TABLE 22.1.2  Topological thresholds for the random 2-complex $Y = Y(n, p)$. c.f. in the TIGHT column means that the bound is best possible up to a constant factor.

<table>
<thead>
<tr>
<th>PROPERTY</th>
<th>THRSHOLD</th>
<th>SHARP</th>
<th>TIGHT</th>
<th>SOURCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$ is not 1-collapsible</td>
<td>$2.455/n$</td>
<td>one-sided</td>
<td>yes</td>
<td>[CCFK12, ALLM13, AL16]</td>
</tr>
<tr>
<td>$H_2(Y, \mathbb{R}) \neq 0$</td>
<td>$2.753/n$</td>
<td>one-sided</td>
<td>yes</td>
<td>[Koz10, LP14, AL15]</td>
</tr>
<tr>
<td>cdim $\pi_1(Y) = 2$</td>
<td>$\Theta(1/n)$</td>
<td>?</td>
<td>c.f.</td>
<td>[CF15a, New16]</td>
</tr>
<tr>
<td>$Y$ is not embeddable in $\mathbb{R}^4$</td>
<td>$\Theta(1/n)$</td>
<td>?</td>
<td>c.f.</td>
<td>[Wag11]</td>
</tr>
<tr>
<td>$Y$ is pure 2-dimensional</td>
<td>$2 \log n/n$</td>
<td>sharp</td>
<td>yes</td>
<td>[LM06]</td>
</tr>
<tr>
<td>$H_1(Y, \mathbb{Z}) = 0$</td>
<td>$2 \log n/n$</td>
<td>sharp</td>
<td>yes</td>
<td>[LM06, MW09]</td>
</tr>
<tr>
<td>$H_1(Y, \mathbb{Z}) = 0$</td>
<td>$2 \log n/n$</td>
<td>sharp</td>
<td>yes</td>
<td>[LM06, HKP12]</td>
</tr>
<tr>
<td>$\pi_1(Y)$ has property (T)</td>
<td>$2 \log n/n$</td>
<td>sharp</td>
<td>yes</td>
<td>[HKP12]</td>
</tr>
<tr>
<td>$H_1(Y, \mathbb{Z}) = 0$</td>
<td>$O(\log n/n)$</td>
<td>sharp</td>
<td>c.f.</td>
<td>[HKP17]</td>
</tr>
<tr>
<td>cdim $\pi_1(Y) = \infty$</td>
<td>$1/n^{3/5}$</td>
<td>coarse</td>
<td>yes</td>
<td>[CF13]</td>
</tr>
<tr>
<td>$Y$ contains arbitrary subdivisions</td>
<td>$\theta(1/\sqrt{n})$</td>
<td>?</td>
<td>c.f.</td>
<td>[GW16]</td>
</tr>
<tr>
<td>$\pi_1(Y) = 0$</td>
<td>$O(1/\sqrt{n})$</td>
<td>?</td>
<td>no</td>
<td>[BHK11, GW16, KPS16]</td>
</tr>
</tbody>
</table>
“I predict a new subject of statistical topology. Rather than count the number of holes, Betti numbers, etc., one will be more interested in the distribution of such objects on noncompact manifolds as one goes out to infinity.”

–Isadore Singer, 2004
Motivation: why stochastic topology?
Randomness models nature
Randomness models nature

- physics: black holes, quantum gravity, etc.
Randomness models nature

- statistics: topological data analysis
Randomness models nature

• mathematics: models for “typical” objects
“Many simplicial complexes that arise in combinatorics are homotopy equivalent to a wedge of spheres. I have often wondered if perhaps there is some deeper explanation for this.”

—Robin Forman
The probabilistic method provides existence proofs.
The probabilistic method provides existence proofs

- Extremal graph theory, Ramsey theory
The probabilistic method provides existence proofs

• Extremal graph theory, Ramsey theory

• Various models of expanders
The probabilistic method provides existence proofs

• Extremal graph theory, Ramsey theory
• Various models of expanders
• Linear algebra
The probabilistic method provides existence proofs

- Extremal graph theory, Ramsey theory
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- Linear algebra
- Geometric group theory
The probabilistic method provides existence proofs

- Extremal graph theory, Ramsey theory
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- Linear algebra
- Geometric group theory
- ...
Random graphs
Define $G(n, p)$ to be a random graph on vertex set $[n] = \{1, 2, \ldots, n\}$, where each edge has probability $p$, independently.
Usually $p = p(n)$ and $n \to \infty$.

We say that event happens with high probability (w.h.p.) if the probability $\to 1$. 
In random graph theory, one is often interested in *thresholds* for various graph properties.
Theorem. (Erdős–Rényi, 1959)

If
\[ p \geq \frac{(1 + \epsilon) \log n}{n} \]
then w.h.p. \( G(n, p) \) is connected.

If
\[ p \leq \frac{(1 - \epsilon) \log n}{n} \]
then w.h.p. \( G(n, p) \) is disconnected.
Theorem. (Erdős–Rényi, 1959)

If
\[ p = \frac{\log n + c}{n} \]

then
\[ \mathbb{P}[G(n, p) \text{ is connected}] \rightarrow e^{-e^{-c}}. \]
At $p = 1/n$,

- giant component appears,
- cycles appear,
- arbitrary topological minors exist,
- $G(n,p)$ is not planar.
At $p = \log n/n$, 

- no isolated vertices,
- $G(n,p)$ becomes connected,
- $G(n,p)$ becomes an expander.
Random 2-complexes
\[ p = \frac{c}{n} \]
\[ p = \frac{c}{n} \]

[CCFK12]

1-collapsible

\[ c \]

\[ 0 \quad 0.5 \quad 2.433 \ldots \quad 2.754 \ldots \]
\[ p = \frac{c}{n} \]

[CKP14]

Giant subcomplex

\[ c \]

\[ 0 \quad 0.5 \quad 2.433 \ldots \quad 2.754 \ldots \]
\[ p = \frac{c}{n} \]

[ALLM13, AL16 ]

not 1-collapsible

\[ c \]

\[ 0 \quad 0.5 \quad 2.433 \ldots \]

\[ 2.754 \ldots \]
\[ p = \frac{c}{n} \]

\[[\text{AL15, LP16}]\]

\[ H_2(Y, \mathbb{R}) \neq 0 \]

c

\[ 0 \quad 0.5 \quad 2.433 \ldots \]
Experiments

$n=75$

<table>
<thead>
<tr>
<th>Faces</th>
<th>$H_2$</th>
<th>$H_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2469</td>
<td>$\mathbb{Z}^4$</td>
<td>$\mathbb{Z}^{236}$</td>
</tr>
<tr>
<td>2470</td>
<td>$\mathbb{Z}^4$</td>
<td>$\mathbb{Z}^{235} \times \mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>2471</td>
<td>$\mathbb{Z}^4$</td>
<td>$\mathbb{Z}^{234} \times \mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>2472</td>
<td>$\mathbb{Z}^4$</td>
<td>$\mathbb{Z}^{233} \times \mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>2473</td>
<td>$\mathbb{Z}^4$</td>
<td>$\mathbb{Z}^{232} \times \mathbb{Z}/2\mathbb{Z}$</td>
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<tr>
<td>2474</td>
<td>$\mathbb{Z}^4$</td>
<td>$\mathbb{Z}^{231} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$</td>
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<td>2475</td>
<td>$\mathbb{Z}^4$</td>
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<tr>
<td>2476</td>
<td>$\mathbb{Z}^5$</td>
<td>$\mathbb{Z}^{230} \times \mathbb{Z}/2\mathbb{Z}$</td>
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<tr>
<td>2477</td>
<td>$\mathbb{Z}^5$</td>
<td>$\mathbb{Z}^{229} \times \mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>2478</td>
<td>$\mathbb{Z}^6$</td>
<td>$\mathbb{Z}^{229}$</td>
</tr>
</tbody>
</table>
Experiments

\[ n=16, \, d=5 \]

<table>
<thead>
<tr>
<th>Faces</th>
<th>( H_5 )</th>
<th>( H_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2972</td>
<td>( \mathbb{Z}^6 )</td>
<td>( \mathbb{Z}^{37} )</td>
</tr>
<tr>
<td>2973</td>
<td>( \mathbb{Z}^6 )</td>
<td>( \mathbb{Z}^{36} \times \mathbb{Z}/1147712621067945810235354141226409657574376675\mathbb{Z} )</td>
</tr>
<tr>
<td>2974</td>
<td>( \mathbb{Z}^7 )</td>
<td>( \mathbb{Z}^{36} )</td>
</tr>
</tbody>
</table>
## Experiments

| Vertices | Trials with nontrivial torsion | Average size of $\ln(|(H_1(X))_T|)$ |
|----------|-------------------------------|-------------------------------------|
| 50       | 1,936                         | 12.4362 ± 4.2007                    |
| 60       | 1,986                         | 28.4761 ± 5.3041                    |
| 70       | 1,998                         | 51.0385 ± 5.9342                    |
| 80       | 1,995                         | 80.0638 ± 7.5880                    |
| 90       | 2,000                         | 115.5589 ± 7.7336                   |
| 100      | 1,998                         | 157.4427 ± 7.5883                   |
## Experiments

<table>
<thead>
<tr>
<th>Group</th>
<th>$n = 50$</th>
<th>$n = 60$</th>
<th>$n = 75$</th>
<th>$n = 100$</th>
<th>$n = 125$</th>
<th>Cohen–Lenstra ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trivial Group</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>0.9045</td>
<td>1.0364</td>
<td>0.9752</td>
<td>0.9965</td>
<td>1.0076</td>
<td>1</td>
</tr>
<tr>
<td>$\mathbb{Z}/4\mathbb{Z}$</td>
<td>1.8822</td>
<td>1.9795</td>
<td>2.0063</td>
<td>2.0314</td>
<td>2.0423</td>
<td>2</td>
</tr>
<tr>
<td>$\mathbb{Z}/8\mathbb{Z}$</td>
<td>3.8530</td>
<td>4.3975</td>
<td>3.8548</td>
<td>4.2812</td>
<td>4.2525</td>
<td>4</td>
</tr>
<tr>
<td>$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$</td>
<td>5.3591</td>
<td>5.6712</td>
<td>6.0705</td>
<td>5.8846</td>
<td>5.9958</td>
<td>6</td>
</tr>
<tr>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/4\mathbb{Z}$</td>
<td>6.4772</td>
<td>7.6464</td>
<td>7.4177</td>
<td>8.2820</td>
<td>7.9560</td>
<td>8</td>
</tr>
<tr>
<td>$\mathbb{Z}/16\mathbb{Z}$</td>
<td>7.2026</td>
<td>7.6666</td>
<td>7.1561</td>
<td>7.3040</td>
<td>7.3316</td>
<td>8</td>
</tr>
<tr>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/8$</td>
<td>13.1756</td>
<td>14.6363</td>
<td>14.9526</td>
<td>17.5120</td>
<td>13.7251</td>
<td>16</td>
</tr>
<tr>
<td>$\mathbb{Z}/32\mathbb{Z}$</td>
<td>15.3466</td>
<td>16.8488</td>
<td>16.2342</td>
<td>19.9109</td>
<td>14.6262</td>
<td>16</td>
</tr>
</tbody>
</table>
Conjecture: at $p = 2.7538/n,$

- there is a torsion burst in $H_1(Y(n, p)),$ and at its peak it is Cohen–Lenstra distributed,
- $Y(n, p)$ is not embeddable in $\mathbb{R}^4,$ and
- $\pi_1(Y(n, p))$ goes from free to not free.
$p = c/n$

another phase transition
At $p = 2 \log n/n$,

- no isolated edges

- $H_1(Y, \mathbb{Z}/\ell) = 0$  \[LM06, MW09]\]

- $H_1(Y, \mathbb{R}) = 0$  \[HKP12]\]

- $H_1(Y, \mathbb{Z}) = 0$  \[LP16, HKP17]\]
Is \( p = d \log n / n \) the threshold for \( H_{d-1}(Y_d(n, p), \mathbb{Z}) = 0 \)?
Random fundamental groups
\[ p = \frac{c}{n} \]
\[ p = \frac{c}{n} \]

[New16]

\( \pi_1(Y) \text{ free} \)

\( 2.433 \ldots \)
\[ p = \frac{c}{n} \]

[New16]

\[ \pi_1(Y) \text{ not free} \]

\[
\begin{align*}
0 & \quad 2.433 \ldots & \quad 2 \log n & \quad \sqrt{n} \\
\end{align*}
\]
\[ p = c/n \]

\[ \pi_1(Y) \text{ is Kazhdan} \]

\[ [\text{HKP12}] \]
\[ p = \frac{c}{n} \]

\[
\pi_1(Y) = 0
\]

[BHK11, GW16, KPS16]
What is the true threshold for $\pi_1(Y(n, p)) = 0$?
What is the true threshold for $\pi_1(Y(n, p)) = 0$?

Conjecture: there is a sharp threshold at $p = c/\sqrt{n}$ for some constant $c > 0$. 
If $2 \log n/n \ll p \ll 1/\sqrt{n}$, are there any nontrivial finite quotients of $\pi_1(Y(n, p))$?
If $2 \log n/n \ll p \ll 1/\sqrt{n}$, are there any nontrivial finite quotients of $\pi_1(Y(n, p))$?

• No maps to any finite abelian groups. [LP16, HKP17]

• No maps to any fixed non-abelian targets. [Mesh17]

• Kazhdan—not many unitary representations. [HKP12]
Thanks for your time and attention!