SPARSE LOCALLY-JAMMED DISK PACKINGS

MATTHEW KAHLE

Abstract. We construct arbitrarily sparse locally-jammed packings of non-overlapping congruent disks in various finite area regions — in particular, we give constructions for the square, hexagon, and for certain flat tori.

1. Introduction

The densest disk packing in the plane was proved to be the hexagonal packing by Axel Thue in 1910 [17], pp. 257–263, although some consider Thue's proof flawed, see Conway & Sloane [7]. Densest disk packing in various bounded regions, triangles, squares, circles, for example is also well studied, and much less is known; see for example the writings of Graham et al [11, 2, 12, 13] and Melissen [15, 14]. See also Brass, Moser, and Pach's book [4] on open problems in discrete geometry.

We use "locally-jammed" to describe a configuration of non-overlapping disks where each disk is held in place by its neighbors (and in the case of a bounded region, by the boundary). In other words, all the angles in the contact graph are strictly less than π. This terminology is used in the physics literature on hard particle packing [18], and these configurations have more often been called "stable" in the discrete geometry literature [16]. The main question we are interested in is: what are the sparsest such configurations?

In 1964 K. Böröczky disproved a conjecture of Fejes Tóth that a locally-jammed arrangement of disks in the plane must have positive density [3]. This counterexample is a main inspiration for the examples given in this article, and our main point is to extend Böröczky's construction to from the unbounded case of the plane to certain finite area regions.

Whereas Böröczky's configuration is zero-density, the best we could hope for in a bounded region is a sequence of locally-jammed configurations with the number of disks $n \to \infty$ and with density approaching zero. If we could attain such sequences, then the interesting question would be how fast can the density approach zero as $n \to \infty$. The examples we construct have $r \leq C/n$ for some constant C depending on the region, which is best possible, up to the constant C.

For certain regions locally-jammed configurations of density $O(1/n)$ are easy to find. Figure 1 shows a locally-jammed configuration in a circle. It is clear that we can find such configurations for arbitrary n and $r = O(1/n)$. The same argument holds in any smoothly bounded 2-dimensional smoothly bounded compact body.

To find such constructions in polygonal regions, seems more difficult, and is the main result of our paper. One consequence is that the Metropolis algorithm for

Date: February 20, 2012.
2010 *Mathematics Subject Classification.* 52C15, 37A25.
Key words and phrases. Disc packing, hard spheres, Boltzmann distribution.
Supported in part by Stanford's NSF-RTG grant in geometry and topology.

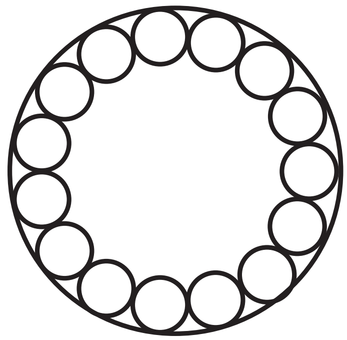

Figure 1. A locally-jammed disk packing in a circle.

hard disks, a well-studied Markov chain for sampling from the configuration space of hard disks, is not necessarily ergodic in the thermodynamic limit for such domains, even for relatively small radius disks. Diaconis, Lebeau, and Michel recently put upper bounds on the mixing time of the Metropolis algorithm for such domains given an assumption that $r = O(1/n)$, and for the regions studied here we see that this assumption is necessary [8, 9].

2. Sparse locally-jammed configurations

A positioning of n non-overlapping disks of radius r in the unit square $[0,1]^2$ is a *configuration.*

We first describe a construction of a one-way infinite "bridge" in the plane due to Böröczky [3]. We follow closely Pach and Sharir's description in [16]. For convenience of notation, all of the disks for now are unit radius; they can be rescaled as necessary later to fit in a box. First, the construction is symmetric about the x-axis, so we only describe disks with their centers on or above the axis. Let $a_1 = (0, 2+\sqrt{3})$, $b_1 = (0, \sqrt{3})$, $c_1 = (1, 0)$, as in Figure 2. Let $f(x)$ be strictly convex function defined for $x \geq 0$, with $f(0) = 2 + \sqrt{3}$ and $\lim_{x\to\infty} f(x) = 2\sqrt{3}$. Let C be the curve $C = \{(x, f(x)) \mid x \geq 0\}$, and let $a_1, a_2, a_3, \ldots$, the unique sequence of distinct points on C such that $d(a_i, a_{i+1}) = 2$ for $i = 1, 2, \ldots$.

Now set b_2 to be the unique point to the right of a_2 such that $d(b_2, a_2) = 2$, and $d(b_2, c_1) = 1$. Then set c_2 to be the point on the x-axis to the right of b_2 with $d(c_2, b_2) = 1$. Inductively define b_{i+1} to be the point to the right of a_i such that $d(b_{i+1}, a_{i+1}) = 2$ and $d(b_{i+1}, c_i) = 2$. Similarly, set c_{i+1} to be the point on the x-axis to the right of b_{i+1} such that $d(c_{i+1}, b_{i+1}) = 2$.

The beginning of this construction is illustrated in the left side of Figure 4. The details that the construction results in a well-defined, locally-jammed configuration (except along the four disks on the left) can be found in Böröczky's original paper [3]. He needed a second idea to get a locally-jammed configuration and disprove Fejes Toth's conjecture. He constructed a junction where three of these bridges could meet and hold all the loose disks on the end in place. Böröczky's junction is shown in Figure 3.

We first use a slight modification of Böröczky's bridge construction, due to Pach and Sharir [16]. For $\epsilon > 0$, replace $f(x)$ above by the strictly convex function

$$f_\epsilon(x) := (1+\epsilon)f(x) - \epsilon f(0).$$

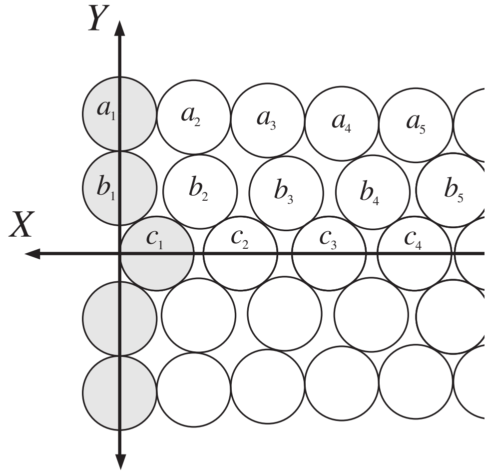

FIGURE 2. Böröczky's bridge. (After Figure 9.2, p. 282, in [16].)

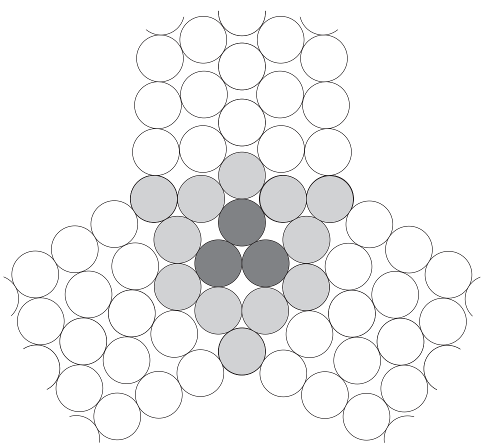

FIGURE 3. Böröczky's junction. The dark shaded disks are locally-jammed, and three copies of the bridge will overlap along the lightly shaded disks. This gives a zero-density locally-jammed configuration in the plane.

We have $f_\epsilon(0) = f(0) = 2 + \sqrt{3}$, and $\lim_{x\to\infty} f_\epsilon(x) < 2\sqrt{3}$, so the sequence no longer continues indefinitely. By the intermediate value theorem, by varying ϵ we can insure that for any arbitrarily large N, some b_N is the last well-defined point, and its x-coordinate is one more than the x-coordinate of a_N.

Then let l be the vertical line through b_N, as in Figure 4, and complete the configuration using l as a line of symmetry, as illustrated. This gives an arbitrarily long symmetric bridge.

The second piece of our construction for the square is the corner "junction" piece in Figure 5. The construction is easily verified to exist. If the center of the bottom

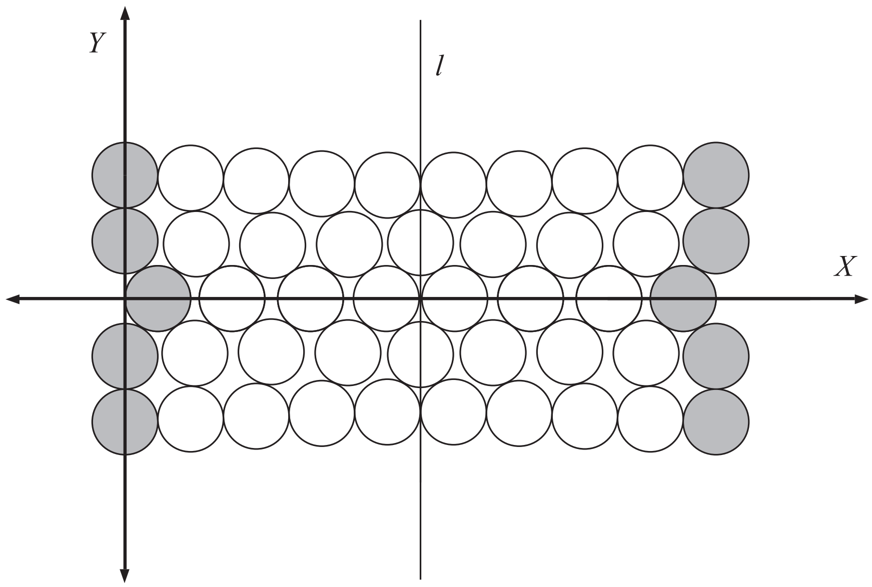

FIGURE 4. Pach and Sharir's bridge.

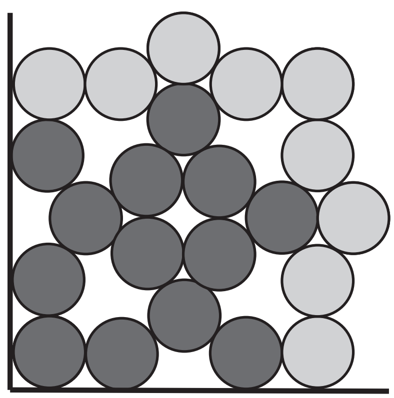

FIGURE 5. The square corner junction.

left disk has coordinates $(0,0)$, then the remaining five disks in the bottom left quarter of the square have coordinates:

$$(0,2), (2,0), (2+\sqrt{3},1), (1,2+\sqrt{3}), \text{ and } (1+\sqrt{3},1+\sqrt{3}).$$

Taking four copies each of the junction and bridge, and arranging them so that they overlap along the lightly-shaded outermost disks as in in Figure 6, the resulting configuration is locally-jammed.

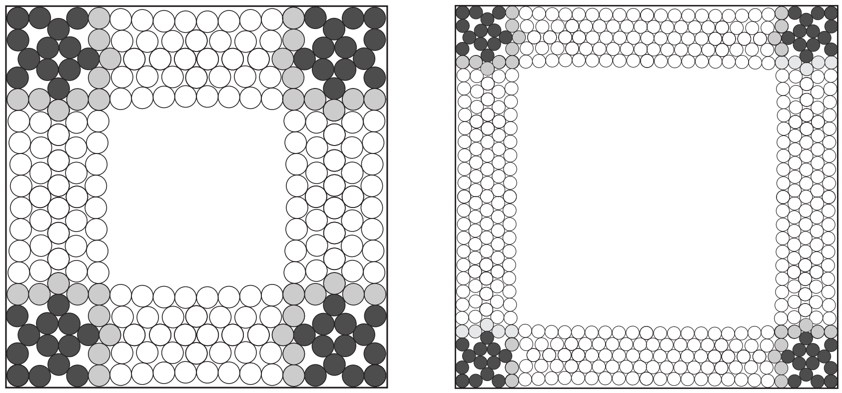

FIGURE 6. By making the bridges arbitrarily long, there are locally-jammed configurations in the square with $r = O(1/n)$.

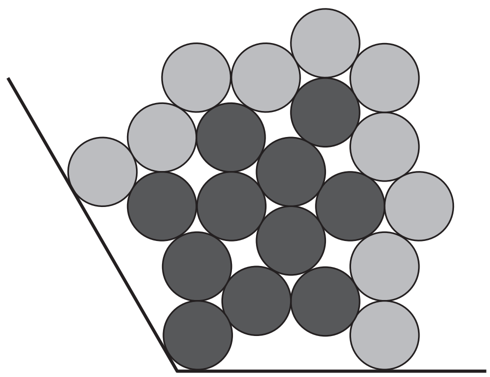

FIGURE 7. Corner junctions for a regular hexagon

By making the bridges arbitrarily long, we have locally-jammed configurations in an square an infinite sequence $\{n_i\}$ disks and $r_i = O(1/n_i)$ as $i \to \infty$. By identifying the edges of the square, we get configurations in a flat square torus.

For similar construction, in Figure 7 we exhibit a corner junction for regular hexagons. It seems likely to us that there are locally-jammed disk packings in any convex polygonal region, and perhaps in any convex region with a piecewise-smooth boundary.

3. DISCUSSION

The configurations discussed in this article are locally-jammed, meaning that no single disk can move, but it is still possible for several disks to move simultaneously, as follows. Connelly noted the following in the context of finding maximally dense disk packings [6].

Lemma 3.1. *Connelly's criterion Suppose $D_1, D_2, \ldots, D_k$ is a collection of hard disks in a convex region with flat boundaries, with disk D_i centered at x_i. Suppose further that $v_1, \ldots, v_k$ are vectors such that*

$$(x_i - x_j) \cdot (v_i - v_j) \geq 0$$

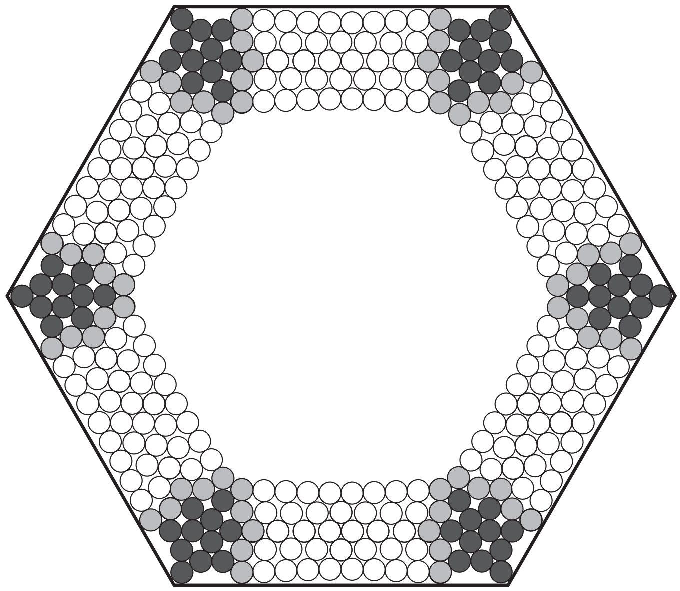

Figure 8. Locally-jammed configurations in a hexagon

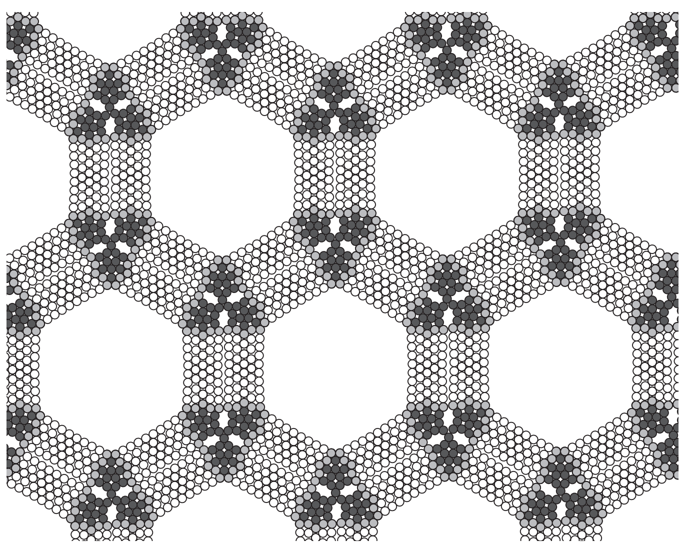

Figure 9. Locally-jammed configuration on a hexagonal flat torus

for every tangent pair of disks $\{D_i, D_j\}$*. Then the disks can simultaneously move, with* D_i *moving in the direction of* v_i*, for some finite time.*

There are 234 disks in Figure 6, and a total of 427 contacts. Since each disk has a priori two degrees of freedom, Connelly's criterion leads to a system of 427 linear inequalities in 468 unknowns. Hence there are nontrivial solutions to this system, corresponding to motions of the disks if they are allowed to collectively move. This distinction is made in the physics literature — this disk packing might be said to be locally-jammed but not collectively-jammed [18]. (This distinction is also made in the mathematical literature — a packing C of convex bodies is called k-stable if every k-element subset of C is fixed by the rest; see Chapter 2, section 3, of the book [4].)

These considerations are closely related to the question of whether the *configuration space* of n disks of radius r is connected [1]. This question is fundamental, e.g. from the point of view of ergodic theory. For a discussion of the topology of configuration spaces of hard disks in the context of statistical mechanics, see [5].

The following questions seem open for 2-dimensional bounded regions or compact Riemannian 2-manifolds without boundary.

(1) How small must $r = r(n)$ to ensure that the configuration space of n disks of radius r is connected? From the example of the circle, sometimes it is necessary to make r fairly small, $r < C/n$ for some constant C. One might guess that for many regions, it is sufficient to make $r < C/\sqrt{n}$ but we are not aware of a single example where someone has proved this.

(2) Similarly, how small can $r = r(n)$ be in a collectively-jammed configuration? It has been conjectured that for flat torus, the sparsest strictly-jammed configuration is the (reinforced) Kagome lattice [10].

Acknowledgements

Thanks to Persi Diaconis for pointing out this problem, and for several helpful conversations.

References

[1] Yuliy Baryshnikov, Peter Bubenik, and Matthew Kahle. Min-type Morse theory for configuration spaces of hard spheres. submitted, arXiv:1108.3061, 2011.

[2] David W. Boll, Jerry Donovan, Ronald L. Graham, and Boris D. Lubachevsky. Improving dense packings of equal disks in a square. *Electron. J. Combin.*, 7:Research Paper 46, 9 pp. (electronic), 2000.

[3] K. Böröczky. Über stabile Kreis- und Kugelsysteme. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 7:79–82, 1964.

[4] Peter Brass, William Moser, and János Pach. *Research problems in discrete geometry.* Springer, New York, 2005.

[5] Gunnar Carlsson, Jackson Gorham, Matthew Kahle, and Jeremy Mason. Computational topology for configuration spaces of hard disks. *Phys. Rev. E*, 85:011303, Jan 2012.

[6] Robert Connelly. Rigidity of packings. *European J. Combin.*, 29(8):1862–1871, 2008.

[7] J.H. Conway, N.J.A. Sloane, and E. Bannai. *Sphere packings, lattices, and groups*, volume 290. Springer Verlag, 1999.

[8] P. Diaconis and G. Lebeau. Micro-local analysis for the metropolis algorithm. *Mathematische Zeitschrift*, 262(2):411–447, 2009.

[9] P. Diaconis, G. Lebeau, and L. Michel. Geometric analysis for the metropolis algorithm on lipschitz domains. *Inventiones Mathematicae*, pages 1–43, 2011.

[10] A. Donev, S. Torquato, F.H. Stillinger, and R. Connelly. Jamming in hard sphere and disk packings. *Journal of applied physics*, 95:989, 2004.

[11] R. L. Graham and B. D. Lubachevsky. Repeated patterns of dense packings of equal disks in a square. *Electron. J. Combin.*, 3(1):Research Paper 16, approx. 17 pp. (electronic), 1996.

[12] B. D. Lubachevsky and R. L. Graham. Curved hexagonal packings of equal disks in a circle. *Discrete Comput. Geom.*, 18(2):179–194, 1997.

[13] Boris D. Lubachevsky, Ron L. Graham, and Frank H. Stillinger. Patterns and structures in disk packings. *Period. Math. Hungar.*, 34(1-2):123–142, 1997. 3rd Geometry Festival: an International Conference on Packings, Coverings and Tilings (Budapest, 1996).

[14] Hans Melissen. Densest packings of eleven congruent circles in a circle. *Geom. Dedicata*, 50(1):15–25, 1994.

[15] J. B. M. Melissen. Optimal packings of eleven equal circles in an equilateral triangle. *Acta Math. Hungar.*, 65(4):389–393, 1994.

[16] János Pach and Micha Sharir. *Combinatorial geometry and its algorithmic applications*, volume 152 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2009. The Alcalá lectures.

[17] Axel Thue. *Selected mathematical papers*. Universitetsforlaget, Oslo, 1977. With an introduction by Carl Ludwig Siegel and a biography by Viggo Brun, Edited by Trygve Nagell, Atle Selberg, Sigmund Selberg, and Knut Thalberg.

[18] S. Torquato and F. H. Stillinger. Jammed hard-particle packings: From Kepler to Bernal and beyond. *Rev. Mod. Phys.*, 82(3):2633–2672, Sep 2010.

Institute for Advanced Study

E-mail address: mkahle@math.ias.edu