MATH 775: GRAPH THEORY & COMBINATORICS I

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1. Enumeration

The standard references for enumerative combinatorics are Richard Stanley's EC1 and EC2 [11, 12].

1.1. Factorials, binomial coefficients, multinomial coefficients.

The first useful fact in combinatorics is that $n! = n \times (n-1) \times \cdots \times 1$ is the number of ways to order n objects. Note that when n is large, a useful approximation is Stirling's formula

$$n! \approx \sqrt{2\pi n}(n/e)^n.$$

The next useful fact is the binomial theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i},$$

where the binomial coefficient $\binom{n}{i}$ is defined by

$$\binom{n}{i} = \frac{n!}{i!}(n-i)!.$$

Note that $0! = 1$ by useful convention.

From the binomial theorem, some useful identities can easily be deduced. For example,

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n,$$

or

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

Exercise 1.1. *Deduce both of these identities without resorting to the binomial theorem.*

A more general version of the binomial theorem is the multinomial theorem. For three variables, for example.

$$(x+y+z)^n = \sum_{a+b+c=n} \frac{n!}{a!b!c!} x^a y^b z^c,$$

Exercise 1.2. *Write down and prove a k-variable multinomial theorem.*

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Note that multinomial coefficients naturally arise in certain types of counting problems. For example, one easily checks that the number of anagrams of the word "bananacabana" is

$$\frac{12!}{6!2!1!3!},$$

where 6, 2, 1, 3 count the number of a's, b's, c's, and n's, respectively.

1.1.1. *Stars and bars.* Suppose that one wants to buy n pieces of candy, and there are k different types of candy. (It is assumed that two pieces of the same type are indistinguishable, and that there are an unlimited number of each type.)

Exercise 1.3. *(Done in class.) Show that there are*

$$\binom{n+k-1}{n}$$

different ways to do this.

Exercise 1.4. *Now suppose that you must get at least one piece of each of type. Show that there are*

$$\binom{n-1}{n-k} = \binom{n-1}{k-1}$$

ways to do this.

1.2. **Partitions of sets and Stirling numbers of the second kind.** Recall that an equivalence relation E on a set R is a subset $E \subset R \times R$ such that

(1) $\{r, r\} \in E$ for every $r \in R$ (reflexive),
(2) if $\{r, s\} \in E$ then $\{s, r\} \in E$ (symmetric) , and
(3) if $\{r, s\} \in E$ and $\{s, t\} \in E$, then $\{r, t\} \in E$ (transitive).

This notation can be cumbersome, so one more often sees $a \sim b$ rather than $\{a, b\}$.

A useful way to think of an equivalence relation on S is a *partition* of S. The parts of the partition are the equivalence classes.

Define $[n] = \{1, 2, \ldots, n\}$. The Stirling numbers (of the second kind), denoted $S(n, k)$ or $\left\{ {n \atop k} \right\}$, are the number of partitions of $[n]$ into k nonempty parts.

For example $S(3, 2) = 3$, because

$$\{1, 2, 3\} = \{1, 2\} \cup \{3\} = \{1, 3\} \cup \{2\} = \{1\} \cup \{2, 3\},$$

and there are no other partitions of [3] into two nonempty parts.

Exercise 1.5. *Find formulas for $S(n, 2)$, $S(n, 3)$, and $S(n, 4)$.*

Exercise 1.6. *Find a general formula for $S(n, n-1)$.*

The notation $\left\{ {n \atop k} \right\}$ suggests an analogy to binomial coefficients $\binom{n}{k}$. Similar to the recursion for binomial coefficieints

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

we have the recursion

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}.$$

Exercise 1.7. *(In class.) Prove this recursion formula.*

Exercise 1.8. *Use the recursion formula and the fact that* $S(n,n) = S(n,0) = 1$ *to compute* $S(n,k)$ *for all* $0 \le k \le n \le 6$.

Exercise 1.9. *Prove that* $\left\{ {n \atop n-k} \right\}$ *is a polynomial in* n *for each fixed* k.

Another important identity is the following.

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)\dots(x-k+1) = x^n.$$

We prove this in class.

Exercise 1.10. *(1) Prove that*

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

In particular, the right-hand side is zero for $k > n$. *(Done in class.)*

(2) Let B_n *be the* n*th Bell number, i.e. the total number of partitions of* $[n]$. *Prove that*

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

1.3. **Partitions of numbers.** A partition of an integer n is a sum

$$n = \lambda_1 + \lambda_2 + \dots + \lambda_k$$

where

$$\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_k \ge 1.$$

Remark 1.11. *Please note the difference: a partition of* $[n]$ *is a set partition, as in the previous section. A partition of* n *is a number partition.*

For example, for $n = 4$ a complete list of partitions is as follows.

$$\begin{aligned} 4 &= 4 \\ &= 3 + 1 \\ &= 2 + 2 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1 \end{aligned}$$

Let $p(n)$ denote the number of partitions of n. By the above, $p(4) = 5$.

A *generating function* for $p(n)$ can be given as follows.

$$\begin{aligned} \sum_{i=1}^{\infty} p(i)x^i &= \prod_{i=1}^{\infty} \frac{1}{1-x^i} \\ &= (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\dots \end{aligned}$$

This can be regarded as a purely formal power series, where all we care about is that the coefficient of x^i is $p(i)$. On the other hand, from the point of view of

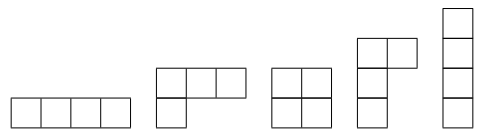

FIGURE 1. The five partitions of $n = 4$

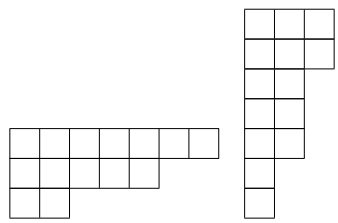

FIGURE 2. A pair of conjugate partitions of $n = 14$

analysis (and analytic number theory) it is worth noting that this series converges for $x \in (-1, 1)$.

Exercise 1.12. *Write a generating function for the number of ways to make n cents with pennies, nickels, dimes, quarters, and half-dollars.*

To compute the partition function $p(n)$ for small values of n, it is useful to have a recursion. Here is it is helpful to introduce an intermediary function $p(k, n)$, the number of partitions of n where each part has size at least k. (So our original function is $p(n) = p(1, n)$.

Noting that $p(k, n) = 0$ if $k > n$ and $p(k, n) = 1$ if $k = n$, we can compute the rest of the values with the recursion

$$p(k, n) = p(k + 1, n) + p(k, n - k).$$

Exercise 1.13. *Why does this recursion hold?*

Exercise 1.14. *Use the recursion formula to compute $p(k, n)$ for $0 \leq k \leq n \leq 10$.*

There are a number of interesting identities involving partitions, and we will discuss a few of the most famous in this section. When discussing partitions, it is sometimes convenient to use Young diagrams, as in Figure 1.

The use of Young diagrams makes the proof of the following almost immediate.

Theorem 1.15. *The number of partitions of n into at most k parts is equal to the number of partitions of n into parts of size at most k,*

Proof. For every partition of n there is a conjugate partition, made by exchanging columns and rows. For example, the partition of $(7, 5, 2)$ is conjugate to the partition $(3, 3, 2, 2, 2, 1, 1)$, as in Figure 2. The result immediate follows. □

Exercise 1.16. *Show that the number of partitions of $n$into no more than k parts is the same as the number of partitions of $n + k$ into exactly k parts.*

Exercise 1.17. *Show that the number of partitions into distinct parts is the same as the number of partitions into odd parts. (This is a theorem of Euler.)*

Let $p_k(n)$ denote the number of partitions of n into exactly k parts.

Exercise 1.18. *Show that the number of partitions of n into at most k parts is equal to $p_k(n+k)$.*

1.3.1. *More on partitions.* Partitions have been studied in number theory for at least a hundred years.

An asymptotic formula for $p(n)$, first due to Hardy and Ramanujan, is given by

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}},$$

as $n \to \infty$.

1.4. **The Twelvefold way.**

> "I will shamelessly tell you what my bottom line is. It is placing balls into boxes, or, as Florence Nightengale David put it with exquisite tact in her book *Combinatorial Chance*, it is the theory of distribution and occupancy."
> — Gian-Carlo Rota, Indiscrete Thoughts [9]

We will describe balls as the set $[n]$ and boxes as the set $[k]$, and putting the balls into boxes as a function $f : [n] \to [k]$.

Restricting to injective, surjective, or arbitrary f gives three possibilities. Whether balls are distinguishable gives two, and whether boxes are distinguishable gives two more. [1]

Multiplying, there are a total of twelve possibilities, and these are what we call the twelvefold way. (Apparently this classification was suggested by Rota, and the name was suggested by Joel Spencer [11].)

1.4.1. *f without restriction.*

(1) Suppose first that balls and boxes are both distinguishable — then there are
$$k^n$$
choices.

(2) Suppose that balls are not distinguishable, but boxes are — then we are back to "stars and bars" of Section 1.1.1, and there are
$$\binom{n+k-1}{n}$$
possibilities. (We only have to count how many balls land in each box...)

(3) Suppose that balls are distinguishable, and boxes are not — then this is a set partition, and there are
$$\sum_{i=0}^{k} \left\{ {n \atop i} \right\}$$
ways to do it.

[1] We could be more precise about what "not distinguishable" means. Formally, we are counting orbits of the natural action of the symmetric group on $[n]$, the symmetric group on $[k]$, and the product of these two groups, on appropriate sets of functions.

(4) Finally, suppose that neither balls nor boxes are distinguishable — then this is a number partition and there are $p_k(n+k)$ ways to do it,by Exercise 1.18.

1.4.2. *Injective f.*

(1) Balls and boxes both distinguishable. There are
$$k(k-1)\dots(k-n+1)$$
ways to do this, by the multiplicative principle. [2]
(2) Balls not distinguishable, but boxes are. Then there are
$$\binom{k}{n}$$
ways. Choose which boxes will contain a ball.
(3) Now if boxes are not distinguishable (whether or not balls are), the case of injective maps becomes simple. There is 1 way if $n \leq k$ and there are 0 ways otherwise.

1.4.3. *Surjective f.*

(1) Balls and boxes both distinguishable. Think of this as a set partition, and then permuting the sets. So there are
$$k! \left\{ {n \atop k} \right\}.$$
(2) Balls not distinguishable, but boxes are. This stars and bars, where you have to have at least one of each type. So there are
$$\binom{n-1}{k-1}$$
ways total.
(3) Balls are distinguishable, but boxes are not. This is exactly the definition of
$$\left\{ {n \atop k} \right\}.$$
(4) Neither balls nor boxes are distinguishable. This is the definition of
$$p_k(n).$$

Now that we have touched on the twelvefold way, we will investigate some special techniques for counting.

1.5. **Catalan numbers.** There are a host of things which Catalan numbers C_n count. See for example the 66 combinatorial interpretations for C_n in exercise 6.19 of [11].

We described several in class, but one is Dyck words of length $2n$.

Definition 1.19. *A Dyck word of length $2n$ has n X's and n Y's, and as you read the word left-to-right, the number of X's is always at least the number of Y's.*

[2]This is sometimes denoted $k^{\underline{n}}$. Note that the expression equals zero for $n > k$.

For example, if $n = 3$ there are 5 Dyck words of length $2n$.

$$XXXYYY$$
$$XXYXYY$$
$$XXYYXY$$
$$XYXXYY$$
$$XYXYXY$$

Exercise 1.20. *Using the Dyck work definition, check that the Catalan numbers satisfy the recurrence*

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Exercise 1.21. *Using that recurrence, derive the formula* $xc^2 + c + 1 = 0$ *for the generating function*

$$c = c(x) = \sum_{i=0}^{\infty} C_i x^i.$$

Exercise 1.22. *From the generating function, derive the formula*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Exercise 1.23. *Derive the formula some other way.*

1.6. **Hook length formula.** The hook length formula counts the number of Standard Young Tableaux of a given shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. We discussed the hook length formula as a sort of multi-dimensional generalization of Catalan numbers.

A beautiful (and very readable) proof of this formula can be found in [4].

2. Graph theory

As we begin our study of graph theory, it is worth mentioning that standard references include the texts by Diestel [3] and by Bollobas [1]. Various versions of the fourth edition of Diestel's book are available online at http://diestel-graph-theory.com/.

2.1. **Eulerian circuits.** The definition of graph is as follows.

Definition 2.1. *A graph is a set of vertices* V *and edges* $E \subseteq \binom{V}{2}$.

If we want to allow directed edges (i.e. $E \subset V \times V$), we emphasize this with the term *digraphs*, and if we want to allow multiple edges between a pair of vertices (i.e. E is now a multiset), we use the term *multigraph*. Sometimes with multigraphs we also allow *loops* (i.e. edges which begin and end at the same vertex).

The degree of a vertex in a multigraph is the number of edges meeting there.

Along the way to solving the Bridges of Konigsberg problem, Euler noticed the following.

Lemma 2.2. *For a finite multigraph (loops allowed) the number of odd-degree vertices is even.*

This allowed him to prove the following.

Theorem 2.3. *A connected multigraph has an Eulerian path if and only if there are* 0 *or* 2 *odd-degree vertices, and an Eulerian circuit if an only if there are* 0 *odd-degree vertices.*

2.2. **Graph coloring.**

Definition 2.4. *A k-coloring of a graph G is a function $V(G) \to [k]$ such that $f(x) \neq f(y)$ whenever $x \neq y$. The chromatic number $\chi(G)$ is the smallest k such that G admits a k-coloring.*

Remark 2.5. *Note that it makes perfect sense to color an infinite graph G with finitely many colors.*

Exercise 2.6. *Prove the Erdős-de Bruijn Theorem: for an infinite graph G,*

$$\chi(G) = \max_{H \subset G,\, H \text{finite}} \chi(H).$$

Your proof should use the well-ordering theorem, or one of its equivalents!

Remark 2.7. *A more general notion is the notion of a graph homomorphism $f : G \to H$. A k-coloring is equivalent to a graph homomorphism $f : G \to K_k$, where K_k is a complete graph on k vertices.*

2.3. **Basic bounds on graph coloring.**

Definition 2.8. *Define the clique number $\kappa(G)$ to be the size of the maximal complete subgraph (clique) of G.*

Clearly $\kappa(G) \leq \chi(G)$.

Definition 2.9. *Define the independence number $\alpha(G)$ to the size of the maximal independent set of vertices in G.*

Exercise 2.10. *Check that*

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

Denote the maximum degree of G by $\Delta(G)$ and the minimum degree by $\delta(G)$.

Exercise 2.11. *Show that $\chi(G) \leq \Delta(G) + 1$.*

The following theorem was proved in class.

Theorem 2.12 (Brooks' Theorem)**.** *If G is connected then $\chi(G) \leq \Delta(G)$ unless G is an odd cycle or a complete graph.*

The proof we discussed in class is apparently due to Lovász [6].

For the next few sections we will concentrate on coloring graphs with special topological or geometric properties.

2.4. **Coloring planar graphs.** A graph that can be drawn in the plane with no edges crossing is called a *planar graph*. Once such a graph is embedded in the plane, it has not only vertices and edges, but faces. [3] One of the most fundamental facts about planar graphs is the following.

[3]By convention there is always an "outer" face, which is unbounded. This may seem more natural if you visualize things on a sphere instead of in the plane.

Theorem 2.13 (Euler formula)**.** *For a finite, connected, planar graph,*

$$v - e + f = 2.$$

It is hard to overestimate the usefulness of the Euler formula.

Exercise 2.14. *Show that for a finite planar graph,*

$$e \leq 3v - 6.$$

Exercise 2.15. *Show that $\delta(G) \leq 5$ for a finite planar graph G.*

Exercise 2.16. *Show that every planar graph is 6-colorable.*

Exercise 2.17. *Show that the complete graph K_5 is not planar. As a corollary, show that every planar graph is 5-colorable.*

Exercise 2.18. *Show that the complete bipartite graph $K_{3,3}$ is not planar.*

By stereographic projection, the Euler formula applies to convex polyhedra as an important special case of planar graphs.

Exercise 2.19. *Suppose that P is a convex polyhedron in $\mathbb{R}^3$, and all its faces are pentagons and hexagons, and three faces meet at every vertex. How many pentagon faces does P have?*

2.5. **Crossing number lemma.** We can refine the dichotomy of graphs as "planar" or "non-planar" in various ways. One way, which we will come back to later, is to consider embeddings of graphs in other surfaces. What we will discuss in this section is the *crossing number*.

In a drawing of a graph $G \to \mathbb{R}^2$, a *crossing* is a pair of edges which intersect anywhere other than at a vertex endpoint.

Definition 2.20. *The crossing number $c(G)$ is the fewest number of crossings over all drawings of G.*

I.e. $c(G) = 0$ if and only if G is planar. It is perhaps surprising that under mild hypotheses $c(G)$ can be bounded from below in terms of the number of vertices and edges alone.

Theorem 2.21 (Crossing number inequality)**.** *If $e \geq 4v$, then*

$$c(G) \geq \frac{e^3}{64v^2}.$$

Exercise 2.22. *Show that Theorem 2.21 gives us a reasonable asymptotic lower bound on crossing numbers of complete graphs $c(K_n)$ — this bound is tight, up to a constant factor, as $n \to \infty$. In other words, the exponents can not be improved.*

Exercise 2.23. *Show that Theorem 2.21 gives us a reasonable asymptotic lower bound on crossing numbers of complete bipartite graphs $c(K_{n,n})$ — this bound is tight, up to a constant factor, as $n \to \infty$.*

Exercise 2.24. *Can you improve the 64 in the denominator if you make a stronger assumption about the number of edges, $e \geq 10v$ or $e \geq 100v$ for example?*

2.6. **Consequences of the crossing number inequality.** Let P be a set of points in the plane, and L a set of lines. Let I denote the set of incidences

$$I = \{(p, l) \in P \times L \mid p \in l.$$

The Szemeredi–Trotter Theorem gives an upper bound on the number of incidences.

Theorem 2.25 (Szemerédi–Trotter)**.**

$$|I| = O\left(|P|^{2/3}|L|^{2/3} + P + L\right).$$

Exercise 2.26. *Give a series of sets of points and lines to show that the exponents* $2/3$ *can not be improved.*

One amazing corollary of Szemerédi–Trotter theorem is to sum-product estimates.

Define the sum-set $A + A = \{x + y \mid x, y \in A\}$. Similarly, define the product set $A \cdot A\{x \cdot y \mid x, y \in A\}$.

For a finite arithmetic progression A, the set $A + A$ is not much larger than A itself. Similarly, for a finite geometric progression A, the set $A \cdot A$ is small. But they can not both be too small. The following can be easily established using Theorem 2.21.

Theorem 2.27. *For any* $\epsilon > 0$*,*

$$\max(|A + A|, |A \cdot A|) \geq C|A|^{4/3}),$$

for some constant C*.*

The best result at the time of this writing may be with replacing the exponent $4/3$ by $3/11$.

Conjecture 2.28 (Erdős–Szemerédi)**.** *For any* $\epsilon > 0$ *and all large enough subsets* $A \subset \mathbb{R}$*,*

$$\max(|A + A|, |A \cdot A|) \geq C_\epsilon |A|^{2-\epsilon}),$$

where C_ϵ *is a constant only depending on* ϵ*.*

(The Erdős–Szemerédi conjecture is actually a bit sharper than this. See http://terrytao.wordpress.com/2007/09/18/the-crossing-number-inequality/ for a proof of Theorem 2.27 and discussion.)

2.7. **Conway's thrackle conjecture.** Conway offers \$1000 for solution to this problem. **Notes to be filled in later.**

2.8. **Coloring graphs on higher genus surfaces.** A basic fact about surfaces is the following.

Theorem 2.29. *Euler characteristic of a surface For an orientable surface of genus* g*, we have*

$$\chi(G) = 2 - 2g.$$

So the Euler characteristic for a sphere is 2, for a torus is 0, etc. This is easy to prove if you accept that Euler characteristic acts like measure and in particular that

$$\chi(A \cup B) = \chi(A) + \chi(B) - A \cap B$$

for compact sets A and B. Then you

Equipped with Euler characteristic alone, one can easily say a lot about coloring graphs on higher genus surfaces.

Exercise 2.30. *Show that $\delta(G) \leq 6$ for a finite graph G embedded in a torus. Conclude that every graph embeddable in a torus is 7-colorable.*

Exercise 2.31. *Prove that this is best possible, by exhibiting an embedding of the complete graph K_7 in the torus.*

Let $\gamma(g)$ denote the smallest number of colors required to color every finite graph embedded in an orientable surface of genus g. A conjecture of Heawood (1890), proved by Ringel and Youngs [8] for $g \geq 1$ is that

$$\gamma(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor.$$

Note that the $g = 0$ case of the Heawood conjecture corresponds to the (much harder) four-color theorem for planar graphs.

The Heawood conjecture can be extended to non-orientable surfaces via the Euler characteristic. If $\gamma(\chi)$ denotes the smallest number of colors required to color every finite graph embedded in an orientable surface of Euler characteristic χ, then

$$\gamma(\chi) = \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor.$$

This holds for all surfaces except the Klein bottle; $\chi = 0$ but all graphs embeddable in the Klein bottle can be six-colored (Franklin, 1934).

Exercise 2.32. *Prove a six-color theorem for the projective plane, and give an embedding of K_6 to show that five colors are not sufficient for some graphs.*

We will come back to the case of the planar graphs.

2.9. **The chromatic number of the plane.** A question of Edward Nelson popularized by Martin Gardner, Paul Erdős, and Alexander Soifer [10], is the following.

Question 2.33. *If we color the plane so that every two points at unit distance receive different colors, what is the fewest number of colors required?*

We can phrase this in terms of graph coloring, as follows.

Definition 2.34. *Define the unit distance graph of the plane $(\mathbb{R}^2, 1)$ with vertices $V = \mathbb{R}^2$ and edges $E = \{(p, q) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid d(p, q) = 1\}$.*

Then Nelson's question is to determine $\chi = \chi\left((\mathbb{R}^2, 1)\right)$

Exercise 2.35. *Show that $4 \leq \chi \leq 7$.*

After over fifty years, these bounds are the best known. It would be a significant achievement to show that $\chi \geq 5$ or $\chi \leq 6$.

One of the stronger results toward improving the lower bound, due to Paul O'Donnell, is that there exist arbitrary girth 4-chromatic unit-distance graphs. (See [10] for an extensive discussion of this and related results.) Even the following special case is seemingly quite difficult.

Exercise 2.36. *Exhibit a triangle-free 4-chromatic unit-distance graph.*

2.10. **Kempe's proof of the 5-color theorem.** Back to planar graphs. We talked about Kempe's proof of the 5-color theorem in class, and I won't reproduce the proof here. This can be found in many combinatorics texts. I recommend [10] for a nice discussion of both the history and the mathematics.

2.11. **Minors and topological minors.** We have so far only dealt with subgraphs, but it is useful to expand our notions of substructures to the more general notions of *minors* and *topological minors*. We follow closely the treatment in Chapter 1 of [3].

A *subdivision* of a graph X is obtained from X by some edges of X with paths. Equivalently, such a graph is obtained by "subdividing" edges of X by adding new vertices. The class of subdivisions of X is denoted TX [4], so we say G *is a* TX as shorthand for G is a subdivision of X.

Definition 2.37. *If Y contains a TX as a subgraph, we say that X is a topological minor of Y.*

We use the terminology "topological minor" because this condition implies that Y contains an embedding of X, when both graphs are considered as topological spaces.

Similarly, replace vertices x of a graph X with connected graphs G_x, and edges $xy \in E(X)$ with non-empty sets of $G_x - G_y$ edges. This yields a graph we call an IX [5].

Definition 2.38. *If Y contains an IX as a subgraph, we say that X is a minor of Y.*

Exercise 2.39. *Minors and topological minors are partial relations on the class of finite graphs.*

Exercise 2.40. *Check that topological minors are minors, but the converse need not hold in general.*

Exercise 2.41. *Prove that if $\Delta(X) \leq 3$ then every IH contains a TH, so in this case X is a minor of G if and only if X is a topological minor of G.*

Minors make it possible to characterize certain graph properties in ways that subgraphs alone can not.

Theorem 2.42 (Kuratowski)**.** *A graph is planar if and only if it contains no K_5 or $K_{3,3}$ topological minors.*

We will prove this in class. A similar theorem, apparently first due to Klaus Wagner is the following.

Theorem 2.43 (Wagner)**.** *A graph is planar if and only if it contains no K_5 or $K_{3,3}$ minors.*

Wagner went deeper and asked if for every minor-closed family of graphs, there is a finite forbidden set. In other words, suppose X is a set of graphs closed under taking minors, i.e. whenever $H \in X$ and G is a minor of H, $G \in X$. Is it true that X can always be defined by a finite forbidden set of minors? (One nice way to say this is the partial order given by minors forms a well-quasi-ordering.)

Exercise 2.44. *Convince yourself that for any surface of genus g, the set of all graphs embeddable in G is closed under taking minors.*

[4] The T stands for 'topological.'

[5] The I stands for 'inflated.'

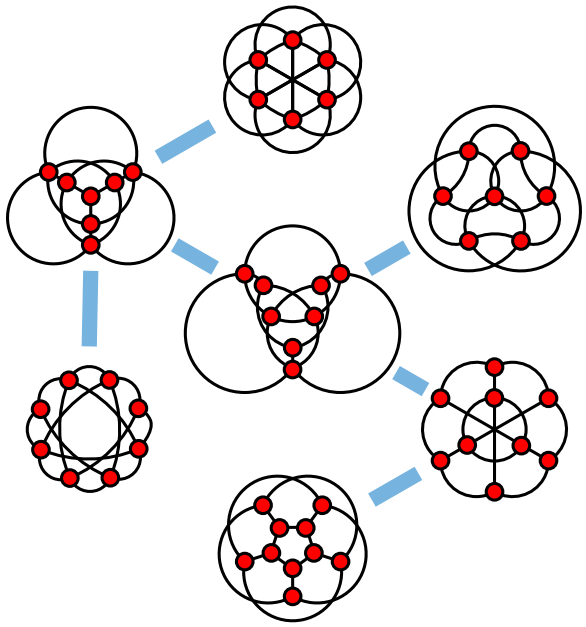

FIGURE 3. The seven forbidden minors for linklessly embeddable graphs. Image by David Eppstein, from the Wikimedia Commons.

It turns out that Wagner's conjecture is true. This called the Minor Theorem, or the Robertson-Seymour theorem after Neil Robertson and Paul Seymour who proved it in a monumental series of papers. This took hundreds of pages and is considered one of the most difficult theorems in combinatorics (if not all of mathematics).

The set of forbidden minors for embedding in the projective plane is known, and there are 35. However, even for the torus there are apparently thousands of forbidden minors and a complete list is still not known.

One interest minor-closed family of for which the forbidden minors have been characterized is linklessly embeddable graphs. Robertson, Seymour, and Thomas showed that every intrinsically-linked graph has a minor isomorphic to one of the graphs in Figure 3.

2.12. **Proof of Kuratowski's and Wagner's theorems.** We will warm up with an easier theorem.

A graph is said to be *outerplanar* if it is planar and can be embedded in the plane with all its vertices on the boundary of the outer face.

Theorem 2.45. *The following are equivalent for a graph G.*

(1) *G is outerplanar;*
(2) *G has no K_4 or $K_{2,3}$ minors;*
(3) *G has no K_4 or $K_{2,3}$ subdivisions.*

Proof. It suffices to consider 2-connected graphs. □

We followed a proof of Kuratowski's theorem in [3].

2.13. **An algebraic characterization of planarity.** Let $V = H_1(G, \mathbb{Z}/2)$ denote first homology of the graph G with $\mathbb{Z}/2$ coefficients. (See for example Hatcher's book.) Note that every $v\ inV$ can be though of as the characteristic function for a set of edges in a natural way.

Theorem 2.46. *A graph is planar if and only if $H_1(G, \mathbb{Z}/2)$ has a sparse basis.*

2.14. **Linklessly embeddable graphs.** We check Conway and Gordan's theorem that K_6 is linklessly embeddable, and briefly discuss the theorem of Robertson, Seymour, and Thomas that characterizes the forbidden minors of linklessly embeddable graphs. (See Figure 3.)

3. Extremal combinatorics

We now turn to the subject of extremal combinatorics.

3.1. Introduction to Ramsey theory.

> "Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6, 6). In that case, he believes, we should attempt to destroy the aliens." — Joel Spencer

Assume that acquaintance is a symmetric relation — i.e. two people are either acquaintances or they are not.

Proposition 3.1. *Given a party of 6 people, there are either 3 mutual acquaintances or 3 mutual non-acquaintances.*

In graph theory we could reformulate this as follows.

Proposition 3.2. *For any 2-coloring of the edges of K_6 there is a monochromatic triangle.*

Exercise 3.3. *Check that this statement is no longer true if 6 is replaced by 5.*

Define the *Ramsey number* $R(s,t)$ to be the smallest N such that: for every 2-coloring of the edges of K_N there exists either a red K_s or a blue K_t.

It is not altogether obvious that such an N exists, but now we will put a bound on N.

Theorem 3.4.

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Noting that $R(1,n) = 1$, and $R(2,n) = n$, and finally that $R(m,n) = R(n,m)$, we have the following corollary.

Corollary 3.5.

$$R(s,t) \leq \binom{s+t}{s}$$

We can use a "probabilistic" argument (or simply counting) to prove the following lower bound on the diagonal Ramsey numbers.

Theorem 3.6.

$$\frac{1}{e\sqrt{2}} n 2^{n/2} < R(n,n)$$

These bounds are easy to come by, but surprisingly little more is known. At least as of 1999, the best bounds are

$$\frac{\sqrt{2}}{e} n 2^{n/2} < R(n,n) < n^{-1/2+c/\sqrt{\log n}} \binom{2n-2}{n-1}.$$

(One nice reference that collects a lot of results like this is the book *Erdős on graphs* [2].)

Using Stirling's formula, for instance, we can write coarser (but simpler) bounds as

$$\left(\sqrt{2}\right)^n < R(n,n) < 4^n,$$

for all $n \geq 3$. To improve either the $\sqrt{2}$ or the 4 are among the most famous unsolved problems in combinatorics.

Another notorious open problem is to give good constructive lower bounds. This is a strange predicament we find ourselves in — that almost all edge colorings of K_N exhibit a certain property, yet we are helpless to give a single example. In fact it is open to give a constructive lower bound of the form $(1+c)^n \leq R(n,n)$ for any $c > 0$.

One of the better constructive lower bounds is by Frankl & Wilson, in Intersection theorems with geometric consequences, *Combinatorica*, 1981. Their result gives that

$$R(n,n) > n^{c \log n / \log\log n}$$

for some constant $c > 0$. So: better than polynomial is obtainable, but no one has reached exponential.

3.2. **More Ramsey theory.**

3.2.1. *Off-diagonal Ramsey numbers.* Things are a little better off the diagonal. It is known for instance that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \frac{n^2}{\log n} \leq R(3,n) \leq c_2 \frac{n^2}{\log n}.$$

The lower bound, by Kim in 1995, is probably one of the best results in Ramsey theory of all time. The order of magnitude for $R(k,n)$ is not known for any fixed $k > 3$.

Exercise 3.7. *Show that for every fixed $k \geq 1$, $R(k,n) = O(n^{k-1})$.*

For larger k it is known that

$$c\left(\frac{n}{\log n}\right)^{(k+1)/2} < R(k,n) < (1+o(1)) \frac{n^{k-1}}{\log^{k-2} n},$$

and it was conjectured by Erdős that

$$R(k,n) \geq \frac{n^{k-1}}{\log^c k},$$

for some constant $c > 0$ and large enough n.

3.2.2. *Multicolor Ramsey numbers.* We can adapt the proof by induction of Theorem 3.5 to establish the existence of multicolor Ramsey numbers.

Let $R(n_1, n_2, \ldots, n_k)$ denote the smallest number N such that whenever the edges of K_N are colored in k colors, there exists a K_{n_i} in color i for some i.

Given that $R(s,t)$ exists, let us see that $R(x,y,z)$ exits. We claim that $R(x,y,z) \leq R(x, R(y,z))$. The proof: use temporary color blindness between the second and third colors.

One multicolor Ramsey number which is known is $R(3,3,3) = 17$. Of course a different kind of asymptotic than the kind discussed above would be to put bounds on $R(3,3,\ldots,3)$, where are k 3's.

3.2.3. *Infinite Ramsey theory.* A basic theorem in infinite Ramsey theory is the following.

Let S be an infinite set, and K_S the complete graph on vertex set S.

Theorem 3.8. *For every k-coloring of the edges of K_S there exists an infinite monochromatic clique.*

We can actually prove a more general theorem, that includes hypergraphs and not only graphs. (In class we followed proof in [3].)

3.2.4. *A detour in infinite combinatorics.* A graph (possibly infinite) is said to be connected if every pair of vertices is connected by a finite path [6].

A *tree* is a connected graph with no cycles. A *spanning tree* of a graph G is a tree with vertex set $V(G)$.

Proposition 3.9. *Every connected graph has a spanning tree.* [7]

The interesting part of the proposition is that it holds for infinite (perhaps uncountable) graphs as well as finite ones.

Proof. Let G be a connected graph and consider the set S of all trees $T \subset G$ ordered by the subgraph relation. We wish to apply Zorn's lemma, so we must check first that every chain $\mathcal{C}$ has an upper bound, i.e. a tree containing every $T \in \mathcal{C}$ as a subgraph. We claim that $T* = \cup\mathcal{C}$ is just such a tree.

Clearly T a subgraph of $T*$ for every $T \in \mathcal{C}$ by construction. The main thing is to check that $T*$ is a tree. Suppose $T*$ is not a tree. Then either $T*$ is disconnected or it has a cycle. If $T*$ is disconnected, then there are two vertices $u, v \in T*$ which are not connected by a path in $T*$. But $u \in T_u$ for some $T_u \in \mathcal{C}$ and $v \in T_v$ for some $T_v \in \mathcal{C}$ and $\mathcal{C}$ is a chain, so either $T_u \subset T_v$ or $T_v \subset T_u$ and in either case u is connected to v by a path in the larger tree, and $T*$ contains this path, a contradiction. The contradiction to $T*$ containing a cycle is similar — every edge in the cycle must appear somewhere in the chain, and then one of the trees in the chain is not acyclic.

Every chain has an upper bound, so S has a maximal element. It must be a spanning tree. □

Another useful fact about connected infinite graphs is the following. Say a graph G is *locally finite* if $\Delta(G) < \infty$.

[6] Compare this to the definition of *path connected* for a topological space.

[7] This is actually equivalent to the Axiom of Choice.

Lemma 3.10 (Konig's infinity lemma)**.** *If G is an infinite, connected, locally finite graph, then G has an infinite long simple path.*

An slightly more general formulation is the following [8]

Lemma 3.11 (Konig's infinity lemma)**.** *Let $V_0, V_1, \dots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex v in a set V_n with $n \geq 1$ has a neighbor $f(v)$ in V_{n-1}. Then G contains a ray $v_0 v_1 \dots$ with $v_n \in V_n$ for all n.*

Our first application of Lemma 3.11 is to prove a countable version of the de Bruijn-Erdős theorem.

Theorem 3.12 (de Bruijn-Erdős theorem)**.** *Let G be a graph on the natural numbers $\mathbb{N}$. If every finite subgraph of G can be k-colored then G can be k-colored.*

This holds (assuming Choice) even for uncountable sets. One nice proof uses Zorn's Lemma (see the graph coloring chapter of [6]), and another uses Tychonoff's compactness theorem.

For another application of Lemma 3.11 we return to Ramsey theory. We show that the infinite version of Ramsey's theorem proves the finite version. (Proof is in [3].)

3.3. **The Erdős-Szekeres problem.** Say that a set of points is in *convex position* if they are all on the boundary of their convex hull. A celebrated early result in combinatorics is the following.

Proposition 3.13. *For every number n there exists an N such that whenever we have N points in $\mathbb{R}^2$ in general position (i.e. no 3 on a line), there are some n of them in convex position.*

Define $g(n)$ to be the smallest such N.

Exercise 3.14. *Show that among any 5 points in $\mathbb{R}^2$, some 4 of them are in convex position, and conclude that $g(4) = 5$.*

Our goal today is to give some upper bounds on Proposition 3.13. Our main reference for this section is the nice survey article [7].

Theorem 3.15.

$$g(n) \leq \min\left(R_4(n,5), R_3(n,n)\right).$$

By $R_k(x,y)$ we mean a certain hypergraph Ramsey number — this is the smallest number N such that in any 2-coloring of the k-sets of $[N]$, there either exists a subset $S \subset [N]$ of size x with all its k-sets blue, or a subset $T \subset [N]$ of size y with all its k-sets red.

Proof. This first argument is due to Erdős and Szekeres, who, unaware of Ramsey's work, rediscovered the existence of hypergraph Ramsey numbers in this geometric context. We check that $g(n) \leq R_4(n,5)$. The thing to notice is that a set S of n points is in convex position if and only if every 4-subset of S is in convex position. Now, given an arrangement of n points in $\mathbb{R}^2$, color each 4-subset blue if it is in convex position and red otherwise. By the definition of $R_4(n,5)$, if $N \geq R_4(n,5)$

[8] This is the formulation in [3]. Note that the "choice" function f is build in here. It may be more natural to not assume the existence of f, but to accept the Axiom of Choice.

we know something about coloring 4-subsets of N. In particular there is either a subset of size n with all 4-subsets colored blue, which corresponds to a set of n points in convex position by the above observation, or a subset of 5 with all its 4-subsets colored red, which is impossible by Exercise 3.14.

Now we check that $g(n) \leq R_3(n,n)$ via an ingenious parity argument due to Scott Johnson [5]. For a finite set of points X, color every 3-subset in $\binom{X}{3}$ according to how many points of X are in the interior of the convex hull of X — if it is an even number of points in the interior, color the triple blue; if it is odd, color it red.

Denote the number of points of X in the convex hull of abc by X_{abc}. For any 4-tuple of points a, b, c, d it is straightforward to check that they are in convex position if and only if

$$X_{abc} + X_{abd} + X_{acd} + X_{bcd} \equiv 0 (\text{mod } 2).$$

Then once we find a set of points of size n with all its triples colored the same way, all their 4-tuples are in convex position, and they are again in convex position. This gives $g(n) \leq R_3(n,n)$, as desired. □

These arguments are quite nice, but unfortunately the upper bounds are not that strong. It seems that $R_3(n,n) \leq R_4(n,5)$ for large n, and it is known that

$$2^{bn^2} \leq R_3 n, n \leq 2^{c^n}$$

for some constants b, c, so at best Theorem 3.15 gives a super-exponential upper bound on $g(n)$. One can do better, and in fact Erdős and Szekeres did do better with a second argument.

Theorem 3.16.

$$g(n) \leq \binom{2n-4}{n-2} + 1.$$

This is via the "caps and cups" argument which we will give in class, but which can also be found in the survey [7]. Clearly this gives an exponential upper bound, of at worst $g(n) = O(4^n)$. After over 75 years, there has still been no improvement to $g(n) = O((4-\epsilon)^n)$ for any $\epsilon > 0$.

The best lower bound (given by a recursive construction) is $g(n) \geq 2^{n-2} + 1$, and Erdős and Szekeres conjectured that this is actually the right answer.

Conjecture 3.17 (Erdős — Szekeres)**.**

$$g(n) = 2^{n-2} + 1$$

They made this conjecture based on remarkably little evidence. But it is known that $g(5) = 9$, and in computer-aided work by Peters and Szekeres, it was apparently established that $g(6) = 17$.

3.4. **Turán's theorem.** A seminal result in extremal graph theory is Turán theory. Let $T_r(n)$ denote the Turán graph — i.e. the complete r-partite graph on n vertices, with parts as equal as possible, and let $t_r(n)$ denote the number of edges in $T_r(n)$.

For any graph H let $\text{ex}(n; H)$ denote the maximum number of edges in a graph with n vertices and no H-subgraph.

Theorem 3.18 (Turán's theorem)**.** *For $r \geq 2$,*

$$ex(n; K_r) = t_r(n) \leq \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right).$$

There are many proofs of Turán's theorem, three of which we covered in class. As Gil Kalai once said, "it is hard *not* to prove Turán's theorem."

Exercise 3.19. *For every fixed $r > 2$ show that $t_r(n) = \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right)$ for infinitely many values of n.*

Exercise 3.20. *Show that*

$$\lim_{n\to\infty} \frac{t_r(n)}{n^2} = \frac{1}{2}\left(1 - \frac{1}{r-1}\right).$$

(This does not follow from the previous exercise, unless you also somehow check that the limit exists.)

An important generalization of Turán's theorem is the Erdős–Stone theorem. Let K_r^s denote the complete r-partite graph where each part has size s.

Theorem 3.21 (Erdős–Stone)**.** *For fixed $r \geq 2$, $s \geq 1$,*

$$ex(n; K_r^s) = \frac{n^2}{2}\left(1 - \frac{1}{r-1} + o(1)\right).$$

A beautiful corollary of the Erdős–Stone theorem is the following.

Corollary 3.22. *For every graph H with at least one edge,*

$$ex(n; H) = \frac{n^2}{2}\left(1 - \frac{1}{\chi(H) - 1} + o(1)\right).$$

The proof is immediate. Let $\chi(H) = r$. Since H is not $r-1$-colorable, H is not a subset of $T_{r-1}(n)$ for any s. On the other hand, H is r-colorable, so is a subgraph of K_r^s for some large enough s. Then apply Erdős–Stone.

Then this corollary tells us the order of magnitude of $\mathrm{ex}(n; H)$ whenever H is not bipartite.

3.5. **Turán numbers for bipartite graphs.**

3.5.1. *Trees and forests.* We wish to show that for every tree (or forest) H, $\mathrm{ex}(n; H)$ grows linearly in H.

Exercise 3.23. *Show that a graph of minimum degree d contains every tree on d edges as a subgraph.*

Exercise 3.24. *Show that a graph of average degree D contains a subgraph of minimum degree $\lfloor D/2 \rfloor$.*

Exercise 3.25. *Conclude a bound on the Turán numbers of trees.* [9]

3.5.2. *Complete bipartite graphs.* Let us put a lower bound on $\mathrm{ex}\,(n; K_{r,s})$ with the probabilistic method.

Consider the edge-independent random graph $G(n, p)$ with n vertices and edge probability p. Let $p = n^{-\alpha}$ where $\alpha > 0$ is to be chosen later.

[9] Erdős and Sos conjectured the right constant in front, but I believe this is still open in general.

What is the expected number of $K_{r,s}$ subgraphs? Clearly

$$\begin{aligned} E[\#K_{r,s}] &= C\binom{n}{r+s}p^{rs} \\ &\approx \frac{C}{(r+s)!}n^{r+s}n^{-\alpha rs}, \end{aligned}$$

so if $r+s-\alpha rs = 0$ then this number tends to a constant. Set $\alpha = (r+s)/rs$. If $p = cn^{-alpha}$ for some sufficiently small $c > 0$ then with probability $> 1/2$ there are no $K_{r,s} subgraphs$. On the other hand with probability $> 1/2$ there are at least $(1-\epsilon)np \approx n^{2-\alpha}$ edges [10].

So there are $K_{r,s}$-free graphs on n vertices with at least $\Omega\left(n^{2-(r+s)/rs}\right)$ edges. In the special case that $r = s$ this gives that

$$\mathrm{ex}\left(n; K_{r,r}\right) = \Omega\left(2 - 2/r\right).$$

We can do a little better with the deletion method.

Exercise 3.26. *Show that*

$$ex\left(n; K_{r,r}\right) = \Omega\left(2 - 2/(r+1)\right).$$

With a simple counting argument we can prove an upper bound.

Exercise 3.27. *Show that*

$$ex\left(n; K_{r,r}\right) = O\left(2 - 1/(r+1)\right).$$

3.5.3. *Constructions.* Here is a construction for graphs with no $K_{2,2} = C_4$-subgraphs.

Let p be a prime and F_p the field of p elements. Define a graph G with vertex set $(F_p)^2 - (0,0)$ and edges $\{(a,b),(x,y)\}$ whenever $ax + by = 1$.

Exercise 3.28. *Clearly $|V(G)| = p^2 - 1$. What is $|E(G)|$? Hint: the line $ax+by = 1$ has exactly p points.*

Exercise 3.29. *Show that G has no C_4 subgraphs.*

A construction with no $K_{3,3}$-subgraphs

3.6. **Set systems with restricted intersections.** Let $\mathcal{F}$ be a family of subsets of $[n]$. We can ask how large can $|\mathcal{F}|$ be given various restrictions on sizes of members of $\mathcal{F}$ and on sizes of their intersections.

Question 3.30. *Suppose that for every set $I \in \mathcal{F}$, $|I|$ is odd and for every pair of sets $I, J \in \mathcal{F}$ we have $|I \cap J|$ is even. How large can $\mathcal{F}$ be?*

Question 3.31. *Suppose that for every set $I \in \mathcal{F}$, $|I|$ is even and for every pair of sets $I, J \in \mathcal{F}$ we have $|I \cap J|$ is even. How large can $|\mathcal{F}|$ be?*

These questions sound very similar, we will see that their answers are very different.

Proposition 3.32. *For Question 3.30, $|\mathcal{F}| \leq n$.*

Proposition 3.33. *For Question 3.31, $|\mathcal{F}| \leq 2^{\lfloor n/2 \rfloor}$.*

[10] The probability is actually easily seen to be $1 - o(1)$ by the Second Moment Method.

Exercise 3.34. *Show that both these Propositions are best possible by giving examples of $\mathcal{F}$ of maximum size.*

The proof of Proposition 3.32 is straightforward.

Exercise 3.35. *Prove Proposition 3.32. (Hint: linear algebra over $\mathbb{Z}_2$.)*

The proof of Proposition 3.34 is also based on linear algebra over $\mathbb{Z}_2$, but is a bit trickier. It is helpful to show that the maximum dimension of a totally isotropic subspace of $(\mathcal{F}_2)^n$ has dimension at most $\lfloor n/2 \rfloor$.

3.7. **A nonuniform modular RW theorem.** Our main result on set systems with restricted intersections was the following.

Theorem 3.36. *Let p be prime, L a set of congruence classes $\pmod p$ with $|L| = s$, and $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ a set of subsets of $[n]$ such that*

(1) $|A_i| \notin L \pmod p$ *for* $1 \le i \le m$, *and*
(2) $|A_i \cap A_j| \in L \pmod p$ *for* $1 \le i < j \le m$.

Then

$$|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}.$$

The proof involves linear algebra over *polynomials* over the finite field $\mathbb{F}_p$.

Exercise 3.37. *Check that if $s \le \lfloor n/4 \rfloor$ then*

$$\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0} \le 2\binom{n}{s}.$$

The bound $2\binom{n}{s}$ is easier to work with, and is good enough for many applications.

Exercise 3.38. *Let $\mathcal{F}$ be a $(2p-1)$-uniform family of subset of $[4p-1]$. Show that if no two members of $\mathcal{F}$ intersect in precisely $p-1$ elements, then*

$$\mathcal{F} \le 2 \cdot \binom{4p-1}{p-1} < 1.7548^{4p-1}.$$

Exercise 3.39. *Let $\chi(\mathbb{R}^n)$ be the chromatic number of n-dimensional Euclidean space with unit distances. Show that for some constant $\epsilon > 0$ and all sufficiently large n,*

$$\chi(\mathbb{R}^n) > (1+\epsilon)^n.$$

(This follows quickly from the previous exercise, if you also quote something about gaps between primes. One useful fact is that the nth gap between primes $g_n = p_{n+1} - p_n$ satisfies $g_n = o(p_n)$. This follows from the Prime Number Theorem and is often good enough to extend coarse combinatorial results on primes to all sufficiently large integers.)

3.8. **A counterexample to Borsuk's conjecture.** We discussed in class a counterexample to Borsuk's conjecture. This is a wonderful and short paper by Kahn and Kalai, almost completely self-contained. However Kahn and Kalai's proof relied on previous work on set systems with restricted intersections of Frankl and Wilson.

The counterexample to Borsuk's conjecture we gave in class was in dimension > 2000. We followed closely the argument in Babai and Frankl's *Linear algebra methods in combinatorics, with applications to geometry and computer science*, using Exercise 3.38 in an essential way.

4. Szemerédi regularity lemma and applications

4.1. **Statement of theorem and proof.** We follow closely the treatment in Alon & Spencer *The Probabilistic Method*, Third Edition.

4.2. **An application: the triangle removal lemma.** Our first application of the Regularity Lemma is the following.

Lemma 4.1. *(Triangle removal lemma) For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that, for any graph G on n vertices with at most δn^3 triangles, it may be made triangle-free by removing at most ϵn^2 edges.*

We proved the triangle removal lemma in class, closely following Alon & Spencer. We note that Jacob Fox recently improved the bounds on δ with a new proof of the triangle removal lemma (more generally, subgraph removal lemma), "A new proof of the subgraph removal lemma," published in Annals of Mathematics (2011).

4.3. **The triangle removal lemma implies Roth's theorem.** A famous theorem of Roth (who received a Fields medal in 1958) is the following.

Theorem 4.2. *(Roth, 1955) For all $\delta > 0$ there exists $N = N(\delta)$ such that, for $n \geq N$, every subset of A of $[n]$ with at least δn elements contains an arithmetic progression of length 3.*

Roth's proof used techniques of analytic number theory, namely an adapted form of the Hardy–Littlewood circle method. We can give a purely combinatorial proof using Lemma 4.1. We will actually prove a stronger result called the corners theorem, and then show that the corners theorem implies Roth's theorem.

Theorem 4.3. *(Corners theorem) Let $\delta > 0$. Then there exists N such that for $n \geq N$, every subset $A \subset [n]^2$ with $|A| \geq \delta n^2$ contains a triple of the form (x, y), $(x + d, y)$, $(x, y + d)$ with $d > 0$.*

Proof. (Proof of Theorem 4.3)

Consider the set $A+A := \{x+y \mid x, y \in A\}$. Since $A \subset [n]$, clearly $A+A \subset [2n]$. We assume $|A| \geq \delta n^2$, therefore there exists some $z \in A + A$ represented as $x + y$ in at least

$$\frac{\left(\delta n^2\right)^2}{(2n)^2} = \frac{\delta^2 n^2}{4}$$

different ways. Define $A' := A \cap (z - A)$ where $z - A = \{z - a \mid a \in A\}$, and set $\delta' = \delta^2/4$. We have $|A'| \geq \delta' n^2$ and if A' contains a triple of the form (x, y), $(x + d, y)$, and $(x, y + d)$ with $d < 0$ then so does $z - A$. Therefore A will contain such a triple with $d > 0$. We may therefore forget about the constraint that $d > 0$ and concentrate on finding a a non-trivial triple with $d \neq 0$.

Define a tripartite graph on vertex set $X \amalg Y \amalg Z$ where $X = Y = [n]$ and $Z = [2n]$. We form a graph by joining $x \in X$ to $y \in Y$ if $(x, y) \in A$, x to z if $(x, z - x) \in A$, and y to z if $(z - y, y) \in A$.

If there is a triangle xyz in G, then (x, y), $(x, y + d)$, and $(x + d, y)$ will all be in A, where $d = z - x - y$ and we will have the required triple, unless $x + y = z$ in which case $d = 0$. Call a triangle xyz in the G *degenerate* if $x + y = z$. Clearly every point of A determines a degenerate triangle, and it is easy to see that these triangles are all edge disjoint. Therefore one can not remove less than δn^2 edges and remove all the triangles in G.

It follows from Lemma 4.1 that there are at least $c(\delta)n^3$ triangles in G. If n is sufficiently large, then this is greater than n^2, and there is at least one non-degenerate triangle in G, corresponding to a corner in A. □

The corners theorem was first proved by Ajtai and Szemerédi. The simpler proof we gave, based on the triangle removal lemma, is due to Solymosi. Next we use the corners theorem to prove Roth's theorem — this proof of Roth's theorem seems to be due to Rusza and Szemerédi.

Proof. (Proof of Theorem 4.2) Let $A \subset [n]$ with $|A| \geq \delta n$, and let $B \subset [2n]^2$ be defined by $B := \{(x, y) \mid x - y \in A\}$. Then

$$|B| \geq \delta n^2 = \frac{\delta}{4}(2n)^2,$$

so for n sufficiently large we must have a corner (x, y), $(x, y + d)$, and $(x + d, y)$ in B. This translates to $x - y - d$, $x - y$, and $x - y + d$ all being in A, an arithmetic progression of length 3. □

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