Introduction to Garland's method, Equivariant Cohomology, Property (T)

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Abstract. Short introduction to Garland's method for vanishing of cohomology based of spectral properties. Along the way, I tried to give some introduction to Equivariant cohomology and property (T).

This note is aimed to present Garland's method for cohomology vanishing based on the eigenvalues of the Laplacians on the links. It basically follows [BŚ97] (although we use a slightly different weight function that is a little more convenient in computations). All mistakes in this note are the fault of the author.

1 Equivariant cohomology

Throughout this note, let Γ be a discrete, finitely generated group (everything I describe below can be done for a more general class of groups, but for our proposes this assumption will do).

Definition 1.1. *Given a set S, a group action of Γ on X is a map*

$$\rho : \Gamma \to Auto(S),$$

such that ρ is a homomorphism. When there is no ambiguity, we denote $\rho(g).x = g.x$.

Important examples:

1. $S = X$ is a simplicial complex and Γ acts simplicially.
2. $S = A$ is an Abelian group and $\forall a, b \in A, g.(a + b) = g.a + g.b$.
3. $S = H$ is a complex (or real) Hilbert space and ρ is a representation of Γ is $\mathcal{U}(H)$ (or into $\mathcal{O}(H)$).

Remark 1.2. *Given a group Γ and S with any of the above structures, there is usually more than one way that Γ can act on S.*

Next, we want to introduce equivariant cohomology in the following set up:

1. Γ as above.
2. H a real Hilbert space and Γ acts on H as above. Denote the action by ρ (you can basically define the same way for any A Abelian, but we are interested in real Hilbert spaces).
3. X is a locally finite simplicial complex and Γ acts on X cocompactly and freely (i.e., X/Γ is finite and the stabilizer of any vertex in X is the trivial group). We'll assume this action is fixed (we will talk about different actions on H).

Define as follows:

1. For $-1 \leq k \leq d$, $\Sigma(k)$ are the ordered k-simplices of X.
2. For $-1 \leq k \leq d$, $\Sigma(k, \Gamma) = \Sigma(k)/\Gamma$ are orbit representatives.
3. For $-1 \leq k \leq d$, $C^k(X, \rho)$ are the equivariant k-cochains into H: i.e., $\phi \in C^k(X, \rho)$ is a map $\phi : \Sigma(k) \to H$ such that:

$$\forall (v_0, ..., v_k) \in \Sigma(k), \pi \in Sym\{0, ..., k\}, \phi((v_0, ..., v_k)) = sign(\pi)\phi((v_{\pi(0)}, ..., v_{\pi(k)})),$$

$$\forall (v_0, ..., v_k) \in \Sigma(k), \forall g \in \Gamma, \rho(g).\phi((v_0, ..., v_k)) = \phi(g.(v_0, ..., v_k)).$$

After the above, define the differential $d_k : C^k(X, \rho) \to C^{k+1}(X, \rho)$ in the usual way:

$$d\phi((v_0, ..., v_{k+1})) = \sum_{i=0}^{k+1} (-1)^i \phi((v_0, ..., \widehat{v_i}, ..., v_{k+1})).$$

(Check - $d_{k-1}d_k = 0$). The equivariant cohomology is then defined in the usual way:

$$B^k(X, \rho) = im(d_{k-1}), Z^k(X, \rho) = ker(d_k), \widetilde{H}^k(X, \rho) = \frac{Z^k(X, \rho)}{B^k(X, \rho)}.$$

We note two extremities:

1. X **is contractible** (i.e., X doesn't have any cohomology of its own): then $\widetilde{H}^k(X, \rho)$ depends only on Γ and we denote $\widetilde{H}^k(\Gamma, \rho)$ - this is called the group cohomology (w.r.t ρ). We remark that for every Γ one can arrange X contractible on which it acts freely and cocompactly, so $\widetilde{H}^k(\Gamma, \rho)$ is always defined - this is done be making X the "simplex" spanned by Γ, i.e., the vertices of X are elements of Γ and every $(k+1)$-tuple of vertices span a k-dimensional simplex (further details on group cohomology in general can be found in [Bro94]).
2. Γ **is trivial (or ρ is the trivial action)**: then $\widetilde{H}^k(X, \rho) = \widetilde{H}^k(X, H)$.

In general, $\widetilde{H}^k(X,\rho)$ is some sort of mix of all the $\widetilde{H}^i(X,H)$'s and $\widetilde{H}^i(\Gamma,\rho)$'s where $-1 \leq i \leq k$ (see exact details in [Bro94] in the chapter about equivariant cohomology).

Definition 1.3. *Γ has property (T) if **for every** unitary representation ρ of Γ on a complex Hilbert space, we have $\widetilde{H}^1(\Gamma,\rho) = 0$.*
Property (T) is equivalent (in a non obvious way) to the following property call (FH): Γ has property (FH) if every action of Γ on a real Hilbert has a fixed point.
The proof of this equivalence and a much more comprehensive account on property (T) can be found in the (great) book - [BdlHV08].

Next, we'll develop technical tools to show vanishing of cohomology.

Definition 1.4. *function $m : \bigcup_{k=-1}^{d} \Sigma(k) \to \mathbb{R}_+$ to be a weight function if:*

$$\forall \sigma \in \Sigma(d), m(\sigma) > 0, \ \textit{m doesn't chage with reordering.}$$

$$\forall -1 \leq k \leq d-1, \forall \tau \in \Sigma(k), m(\tau) = \frac{1}{(k+2)!} \sum_{\sigma \in \Sigma(k+1), \tau \subset \sigma} m(\sigma),$$

where $\tau \subset \sigma$ does not refer to the ordering only to inclusion as sets.

Example 1.5. *one can take $m(\sigma) = 1$ for every $\sigma \in \Sigma(d)$ and get for $\tau \in \Sigma(k)$,*

$$m(\tau) = (d-k)! \, |\{\sigma \in X^{(d)} : \tau \subset \sigma\}|.$$

This is called the homogeneous weight function.

Using m, we'll make $C^k(X,\rho)$ into a real Hilbert space, by introducing the following inner-product: for every $\phi, \psi \in C^k(X,\rho)$

$$\langle \phi, \psi \rangle = \sum_{\tau \in \Sigma(k,\Gamma)} \frac{m(\tau)}{(k+1)!} \langle \phi(\tau), \psi(\tau) \rangle.$$

Notation: we'll use $\|.\|$ to denote the norm in C^k and $|.|$ to denote the norm in H.

Remark 1.6. *With the above inner-product $C^k(X,\rho)$ is a Hilbert space and $B^k(X,\rho), Z^k(X,\rho)$ are subspaces of it. Note that the quotient of the two (i.e., $\widetilde{H}^k(X,\rho)$) is not necessarily a Hilbert space, since $B^k(X,\rho)$ is not necessarily closed. Therefore one can define the reduced cohomology:*

$$\widetilde{H}^k_{reduced}(X,\rho) = Z^k(X,\rho)/\overline{B^k(X,\rho)}.$$

With this inner-product, $C^k(X,\rho)$ is a Hilbert space and $d_k : C^k(X,\rho) \to C^{k+1}(X,\rho)$ is a map between Hilbert spaces. Therefore we can define the adjoint of d_k:

$$d_k^* : C^{k+1}(X,\rho) \to C^k(X,\rho).$$

Define the Laplacians as

$$\Delta_k^+ = d_k^* d_k, \Delta_k^- = d_{k-1} d_{k-1}^* : C^k(X,\rho) \to C^k(X,\rho).$$

Also define $\Delta_k = \Delta_k^+ + \Delta_k^-$. Ex. - $\Delta_k^+, \Delta_k^-, \Delta_k$ are all bounded. Notice that for every $\phi \in C^k(X, \rho)$,

$$\langle \Delta_k^+ \phi, \phi \rangle = \langle d_k^* d_k \phi, \phi \rangle = \langle d\phi, d\phi \rangle = \|d\phi\|^2,$$

therefore Δ_k^+ is a positive operator and $ker(\Delta_k^+) = ker(d_k)$. From the same reasoning, Δ_k^- is a positive operator and $ker(\Delta_k^-) = ker(d_{k-1}^*)$. Note that $\overline{im(d_{k-1})} = ker(d_{k-1}^*)^\perp$ and therefore

$$\begin{aligned}
\widetilde{H}^k_{reduced}(X, \rho) &= ker(d_k)/\overline{im(d_{k-1})} \\
&= ker(d_k) \cap (ker(d_{k-1}^*)^\perp)^\perp \\
&= ker(d_k) \cap ker(d_{k-1}^*) \\
&= ker(\Delta_k^+) \cap ker(\Delta_k^-) \\
&= ker(\Delta_k).
\end{aligned}$$

Therefore $\widetilde{H}^k_{reduced}(X, \rho) = 0$ if and only if $ker(\Delta_k) = \{0\}$.
Moreover, if one can also show that there is $\varepsilon >$ such that for every $\phi \in ker(d_k)$,

$$\langle \Delta_k^- \phi, \phi \rangle \geq \varepsilon \|\phi\|^2,$$

then $\widetilde{H}^k(X, \rho) = 0$. Explanation:
First, note that the above condition implies that $\widetilde{H}^k_{reduced}(X, \rho) = 0$. Second, note that by this condition the image of Δ_k^- is closed in $ker(d_k)$. But we have that

$$im(\Delta_k^-) \subseteq im(d_{k-1}),$$

$$\overline{im(\Delta_k^-)} = ker(\Delta_k^-)^\perp = ker(d_{k-1}^*)^\perp = \overline{im(d_{k-1})}.$$

Therefore, if $im(\Delta_k^-)$ is closed, then

$$\overline{im(d_{k-1})} = \overline{im(\Delta_k^-)} = im(\Delta_k^-) \subseteq im(d_{k-1}),$$

which shows that $\overline{im(d_{k-1})} = im(d_{k-1})$.
Terminology - the eigen-cochains of Δ_k are called harmonic forms.

Remark 1.7. *It should be noted that the choice of m does not effect in any way on the vanishing (or non vanishing) of the cohomology.*

Proposition 1.8. *For every $-1 \leq k < l \leq d$ and every equivariant function $f : \Sigma(k) \times \Sigma(l) \to \mathbb{R}$ we have that*

$$\sum_{\sigma \in \Sigma(l,\Gamma)} \sum_{\tau \in \Sigma(k), \tau \subset \sigma} f(\sigma, \tau) = \sum_{\tau \in \Sigma(k,\Gamma)} \sum_{\sigma \in \Sigma(l), \tau \subset \sigma} f(\sigma, \tau).$$

Proof. Left an an exercise. □

The above proposition allows us to change the order of summation, which we'll do without explicitly mentioning from now on.

Proposition 1.9. *For every $\phi \in C^k(X,\rho)$, $\forall \tau = (v_0, ..., v_{k-1}) \in \Sigma(k-1)$*

$$d^*_{k-1}\phi(\tau) = \sum_{(v)\in\Sigma(0), v\tau\in\Sigma(k)} \frac{m(v\tau)}{m(\tau)}\phi(v\tau),$$

where $v\tau = (v, v_0, ..., v_{k-1})$. Also, for every $\sigma \in \Sigma(k)$,

$$\Delta^+\phi(\sigma) = \phi(\sigma) - \sum_{(v)\in\Sigma(0), v\sigma\in\Sigma(k+1)} \sum_{i=0}^{k} (-1)^k \frac{m(v\sigma)}{m(\sigma)}\phi(v\sigma_i).$$

Proof. For $(v_0, ..., v_k) = \sigma \in \Sigma(k), \tau \in \Sigma(k-1), \tau \subset \sigma$ denote by $[\sigma : \tau]$ the sign of the permutation $\pi \in Sym\{0, ..., k\}$ that has

$$(v_{\pi(1)}, ..., v_{\pi(k)}) = \tau.$$

Let $\psi \in C^{k-1}(X,\rho), \phi \in C^k(X,\rho)$,

$$\begin{aligned}
\langle d_{k-1}\psi, \phi\rangle &= \sum_{\sigma\in\Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!}\langle \sum_{i=0}^{k}(-1)^i\psi(\sigma_i), \phi(\sigma)\rangle \\
&= \sum_{\sigma\in\Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!}\langle \sum_{i=0}^{k}[\sigma : \sigma_i]\psi(\sigma_i), \phi(\sigma)\rangle \\
&= \sum_{\sigma\in\Sigma(k,\Gamma)} \frac{m(\sigma)}{(k+1)!} \sum_{\tau\in\Sigma(k-1),\tau\subset\sigma} \frac{1}{k!}\langle [\sigma : \tau]\psi(\tau), \phi(\sigma)\rangle \\
&= \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{1}{k!} \sum_{\sigma\in\Sigma(k),\tau\subset\sigma} \frac{m(\sigma)}{(k+1)!}\langle [\sigma : \tau]\psi(\tau), \phi(\sigma)\rangle \\
&= \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{m(\tau)}{k!}\langle \psi(\tau), \sum_{\sigma\in\Sigma(k),\tau\subset\sigma} \frac{[\sigma : \tau]m(\sigma)}{(k+1)!\, m(\tau)}\phi(\sigma)\rangle \\
&= \sum_{\tau\in\Sigma(k-1,\Gamma)} \frac{m(\tau)}{k!}\langle \psi(\tau), \sum_{(v)\in\Sigma(0),v\tau\in\Sigma(k)} \frac{m(v\tau)}{m(\tau)}\phi(v\tau)\rangle.
\end{aligned}$$

The second equality is just a simple calculation now. □

Example 1.10. *assume $H = \mathbb{R}$, $X = (V, E)$ is a finite graph (1-dimensional complex) and Γ is the trivial group and m is the homogeneous weight function. Then for every vertex v, $m(v)$ is just the valency of v. For every $\phi \in C^0(X,\mathbb{R})$ we have*

$$d^*_{-1}\phi(\emptyset) = \sum_{v\in V} \frac{m(v)}{m(\emptyset)}\phi(v).$$

(Note that by definition $m(\emptyset) = \sum_{v\in V} m(v)$ so this is a weighted average). Also

$$\Delta^+_0\phi(v) = \phi(v) - \frac{1}{m(v)} \sum_{u\in V,(u,v)\in E} \phi(u),$$

i.e., Δ^+_0 is the usual graph Laplacian. Note that if m is not the homogeneous weight function, Δ^+_0 is the weighted graph Laplacian.

2 Localization

Let $\tau \in \Sigma(k)$. The link of $\tau = (v_0, ..., v_k)$ is the simplicial complex of dimension $d-k-1$ denoted X_τ defined as

$$X_\tau^{(j)} = \{\{u_0, ..., u_j\} \in X^{(j)} : \{v_0, ..., v_k, u_0, ..., u_j\} \in X^{(k+j+1)}\}.$$

We define

$$\Sigma_\tau(j) = \{\sigma \in \Sigma(j) : \tau\sigma \in \Sigma(k+j+1)\}.$$

$$C^j(X_\tau, \rho) = C^j(X_\tau, H),$$

(this is because the action is free. In case the stabilizer is non trivial, one has to consider the action of it). Define also a weight function m_τ as $m_\tau(\sigma) = m(\tau\sigma)$ and an inner-product using m_τ as before: $\forall \phi, \psi \in C^j(X_\tau, H)$,

$$\langle \psi, \phi \rangle = \sum_{\sigma \in \Sigma(j)} \frac{m_\tau(\sigma)}{(j+1)!} \langle \psi(\sigma), \phi(\sigma) \rangle.$$

Next, we define the main technical tool we'll use - localization. For $\phi \in C^l(X, \rho)$, $\tau \in \Sigma(k)$ with $k < l$, define $\phi_\tau \in C^{l-k-1}(X_\tau, H)$ as

$$\phi_\tau(\sigma) = \phi(\tau\sigma).$$

There are 4 key fact that make localization useful:
Fact 1: $\|\phi\|$ can be computed from localizations:
For every $\phi \in C^l(X, \rho)$ and for every $-1 \leq k < l$,

$$\frac{(l+1)!}{(l-k-1)!} \|\phi\|^2 = \sum_{\tau \in \Sigma(k,\Gamma)} \|\phi_\tau\|^2.$$

Proof.

$$\sum_{\tau \in \Sigma(k,\Gamma)} \|\phi_\tau\|^2 =$$

$$\sum_{\tau \in \Sigma(k,\Gamma)} \sum_{\sigma \in \Sigma_\tau(l-k-1)} \frac{m_\tau(\sigma)}{(l-k-1)!} |\phi_\tau(\sigma)|^2 =$$

$$\sum_{\tau \in \Sigma(k,\Gamma)} \sum_{\sigma \in \Sigma_\tau(l-k-1)} \frac{m(\tau\sigma)}{(l-k-1)!} |\phi(\tau\sigma)|^2 =$$

$$\sum_{\tau \in \Sigma(k,\Gamma)} \frac{(l-k-1)!}{(l+1)!} \sum_{\eta \in \Sigma(l), \tau \subset \eta} \frac{m(\eta)}{(l-k-1)!} |\phi(\eta)|^2 =$$

$$\sum_{\eta \in \Sigma(l,\Gamma)} \frac{m(\eta)}{(l+1)!} |\phi(\eta)|^2 \sum_{\tau \in \Sigma(k), \tau \subset \eta} 1 =$$

$$\|\phi\|^2 \binom{l+1}{k+1} (k+1)! = \|\phi\|^2 \frac{(l+1)!}{(l-k-1)!}.$$

□

Fact 2: $\|d^*_{l-1}\|$ can be computed from localizations:
For every $\phi \in C^l(X,\rho)$ and for every $\tau \in \Sigma(l-1)$, $|d^*_{k-1}\phi(\tau)|= |d^*_{-1}\phi_\tau(\emptyset)|$ and therefore

$$l!\,\|d^*_{l-1}\phi\|^2= \sum_{\tau\in\Sigma(l-1,\Gamma)} \|d^*_{-1}\phi_\tau\|^2.$$

Proof. Recall that $d^*_{k-1}\phi(\tau)$ was calculated explicitly as

$$d^*_{k-1}\phi(\tau) = \sum_{v\in\Sigma(0),v\tau\in\Sigma(l)} \frac{m(v\tau)}{m(\tau)}\phi(v\tau).$$

Also, $\phi_\tau \in C^0(X_\tau, H)$ and therefore by the example we had for graphs, we get

$$d^*_{-1}\phi_\tau(\emptyset) = \sum_{v\in\Sigma_\tau(0)} \frac{m_\tau(v)}{m_\tau(\emptyset)}\phi_\tau(v) = \sum_{v\in\Sigma(0),v\tau\in\Sigma(l)} \frac{m(\tau v)}{m(\tau)}\phi(\tau v).$$

Therefore $d^*_{-1}\phi_\tau(\emptyset)$ and $d^*_{k-1}\phi(\tau)$ are equal up to a sign as needed. To get the equality in norms is now easy:

$$l!\,\|d^*_{l-1}\phi\|= \sum_{\tau\in\Sigma(l-1,\Gamma)} m(\tau)|d^*_{l-1}\phi(\tau)|^2 = \sum_{\tau\in\Sigma(l-1,\Gamma)} m(\tau)|d^*_{-1}\phi_\tau(\emptyset)|^2 =$$
$$\sum_{\tau\in\Sigma(l-1,\Gamma)} m_\tau(\emptyset)|d^*_{-1}\phi_\tau(\emptyset)|^2 = \sum_{\tau\in\Sigma(l-1,\Gamma)} \|d^*_{-1}\phi_\tau\|^2.$$

□

Fact 3: $\|d_l\phi\|$ can be computed from localizations:
For every $\phi \in C^l(X,\rho)$ we have that

$$l!\,\|d_l\phi\|^2= \sum_{\tau\in\Sigma(l-1,\Gamma)} \left(\|d_0\phi_\tau\|^2 - \frac{l}{l+1}\|\phi_\tau\|^2\right).$$

Proof. We'll prove only for $l = 1$ (for $l > 1$ the computations are basically the same, but the writing gets cumbersome). Let $(u,v,w) \in \Sigma(2), \phi \in C^1(X,\rho)$, then we have that

$$|d_1\phi((u,v,w))|^2 = |\phi((v,w)) - \phi((u,w)) + \phi((u,v))|^2 =$$

$$|\phi((u,v)) - \phi((u,w))|^2 + |\phi((v,u)) - \phi((v,w))|^2 + |\phi((w,u)) - \phi((w,v))|^2 -$$
$$\left(|\phi((u,v))|^2 + |\phi((u,w))|^2 + |\phi((v,w))|^2\right) =$$

$$|d_0\phi_u((v,w))|^2 + |d_0\phi_v((u,w))|^2 + |d_0\phi_w((u,v))|^2 -$$
$$\left(|\phi((u,v))|^2 + |\phi((u,w))|^2 + |\phi((v,w))|^2\right) =$$

$$|d_0\phi_u((v,w))|^2 + |d_0\phi_v((u,w))|^2 + |d_0\phi_w((u,v))|^2 -$$
$$\left(|\phi((u,v))|^2 + |\phi((u,w))|^2 + |\phi((v,w))|^2\right).$$

Using the above to calculate the norm of $\|d_1\phi\|^2$ gives:

$$\|d_1\phi\|^2= \sum_{u\in\Sigma(0,\Gamma)} \|d_0\phi_u\|^2 - \|\phi\|^2 = \sum_{u\in\Sigma(0,\Gamma)} \|d_0\phi_u\|^2 - \frac{1}{2}\|\phi_u\|^2.$$

□

Fact 4: Assume that X is connected as a graph and finite. Denote by $\lambda(X)$ the smallest positive eigenvalue of $\Delta_0^+ = d_0^* d_0$ on $C^0(X,\mathbb{R})$. Then for every real Hilbert space H and for every $\phi \in C^0(X,H)$ we have that

$$\|d_0\phi\|^2 \geq \lambda(X)\left(\|\phi\|^2 - \|d_{-1}^*\phi\|^2\right).$$

Proof. First, we'll prove this inequality for $\phi \in C^0(X,\mathbb{R})$. Note that by direct computation we have that $\Delta_0^-\phi$ is a constant function and that

$$\Delta_0^-\phi \equiv \sum_{v\in\Sigma(0)} \frac{m(v)}{m(\emptyset)}\phi(v).$$

Therefore $\Delta_0^-\phi$ is the projection of ϕ on the space of constant functions on $\Sigma(0)$, which is exactly $ker(\Delta_0^+)$. Therefore, for every ϕ we have that

$$\|d_0\phi\|^2 = \langle \Delta_0^+\phi, \phi\rangle \geq \lambda(X)\|(I-\Delta_0^-)\phi\|^2 = \lambda(X)\left(\|\phi\|^2 - \|\Delta_0^-\phi\|^2\right) = \lambda(X)\left(\|\phi\|^2 - \|d_{-1}^*\phi\|^2\right).$$

Next, we'll prove the inequality for $\phi \in C^0(X,\mathbb{R}^m)$ for every $m \in \mathbb{N}$. Notice that in this case Δ_+^0 is a linear self-adjoint operator on a finite dimensional Euclidean space and therefore is has a spectral decomposition with exactly the same eigenvalues as Δ_0^+ on $C^0(X,\mathbb{R})$ (for an eigenfunction in $C^0(X,\mathbb{R}^m)$, each fixed coordinate gives an eigenfunction in $C^0(X,\mathbb{R})$). Therefore, by the same considerations as above the inequality holds.
Last, for $\phi \in C^0(X,H)$ where H is any complex Hilbert space, the image of ϕ is always contained in a finite dimensional Euclidean space (of dimension less than $|X^{(0)}|$) and therefore the same inequality holds. □

Combining all the above facts, we are finally ready to prove Garland's vanishing method:

Theorem 2.1. *Let X,Γ above. Assume that X is pure d-dimensional simplicial complex with some weight function m. Fix $0 \leq l \leq d-1$. Assume that for every $\tau \in \Sigma(l-1)$, X_τ is connected and denote by $\lambda(X_\tau)$ the smallest positive eigenvalue of Δ_0^+ on $C^0(X_\tau,\mathbb{R})$ (note that $\lambda(X_\tau)$ depends on the choice of m). If*

$$\min_{\tau\in\Sigma(l-1)} \lambda(X_\tau) > \frac{l}{l+1},$$

then for every isometric action ρ of Γ on a real Hilbert space we have that $\widetilde{H}^l(X,\rho) = 0$.

Proof. Denote

$$\lambda_l = \min_{\tau\in\Sigma(l-1)} \lambda(X_\tau).$$

By fact 3 above, we have for every $\phi \in C^l(X,\rho)$

$$l!\,\|d_l\phi\|^2 = \sum_{\tau\in\Sigma(l-1,\Gamma)} \left(\|d_0\phi_\tau\|^2 - \frac{l}{l+1}\|\phi_\tau\|^2\right).$$

By fact 4, we have that for every $\tau \in \Sigma(l-1)$,

$$\|d_0\phi_\tau\|^2 \geq \lambda_l\left(\|\phi_\tau\|^2 - \|d_{-1}^*\phi_\tau\|^2\right).$$

Therefore

$$l!\,\|d_l\phi\|^2 = \sum_{\tau\in\Sigma(l-1,\Gamma)} \left(\lambda_l\left(\|\phi_\tau\|^2 - \|d^*_{-1}\phi_\tau\|^2\right) - \frac{l}{l+1}\|\phi_\tau\|^2\right).$$

By fact 2, we have that

$$l!\,\|d_l\phi\|^2 = \left(\sum_{\tau\in\Sigma(l-1,\Gamma)} (\lambda_l - \frac{l}{l+1})\|\phi_\tau\|^2\right) - \lambda_l l!\,\|d^*_{l-1}\phi\|^2.$$

Therefore, by fact 1

$$l!\left(\langle\Delta_l^+\phi,\phi\rangle + \lambda_l\langle\Delta_l^-\phi,\phi\rangle\right) \geq (\lambda_l - \frac{l}{l+1})(l+1)!\,\|\phi\|^2.$$

Therefore, if $\phi \in ker(d_k) = ker(\Delta_k^+)$, we have that

$$l!\,\lambda_l\langle\Delta_l^-\phi,\phi\rangle \geq (\lambda_l - \frac{l}{l+1})(l+1)!\,\|\phi\|^2.$$

From previous discussion, this yields $\widetilde{H}^l(X,\rho) = 0$ for every ρ. □

Corollary 2.2. *For any finite simplicial complex X, for every $d \in \mathbb{N}\cup\{0\}$, if for every $\tau \in \Sigma(d-2)$, X_τ is connected and $\lambda(X_\tau) > \frac{d}{d+1}$ (when the Laplacian is usual normalized graph Laplacian), then $\widetilde{H}^{d-1}(X,\mathbb{R}) = 0$.*

Proof. Use the above theorem with respect to $\Gamma = \{e\}$ and the d-skeleton of X with homogeneous weight. □

The above corollary raises the question - why use weights at all. The answer is the following theorem from [Opp12]:

Theorem 2.3. *Let X be a pure d-dimensional simplicial complex. If all the links of X (including X itself) are connected, and*

$$\min_{\tau\in\Sigma(d-2)} \lambda(X_\tau) > \frac{d}{d+1},$$

then for every $-1 \leq k \leq d-2$ one has

$$\min_{\tau\in\Sigma(k)} \lambda(X_\tau) > \frac{k}{k+1},$$

where $\lambda(X_\tau)$ are computed with respect to the weighted Laplacian.

Corollary 2.4. *Let X be a pure d-dimensional simplicial complex. If all the links of X (including X itself) are connected, and*

$$\min_{\tau\in\Sigma(d-2)} \lambda(X_\tau) > \frac{d}{d+1},$$

then for every $0 \leq k \leq d-1$, $\widetilde{H}^k(X,\mathbb{R}) = 0$.

Corollary 2.5. *Let X be a pure d-dimensional simplicial complex and Γ is a group acting freely (in fact - it is enough to assume that the action is proper) and cocompactly on X. If all the links of X (including X itself) are connected, and*

$$\min_{\tau \in \Sigma(d-2)} \lambda(X_\tau) > \frac{d}{d+1},$$

then for every Hilbert space H and every isometric action of Γ on H we have for every $0 \leq k \leq d-1$, $\widetilde{H}^k(X, \rho) = 0$. If X is contractible (or aspherical), then also $\widetilde{H}^k(\Gamma, \rho) = 0$ for every $0 \leq k \leq d-1$.

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