Math 6501: Graph theory & combinatorics I

Matthew Kahle, Ohio State University

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1 Enumeration

Standard references for enumerative combinatorics include Richard Stanley's encyclopedic [3] and [4].

1.1 Factorial

Define the "factorial" $n!$ inductively by $0! = 1$ and $n! = n \times (n-1)!$. The factorial is a fundamental quantity in combinatorics. For example, $n!$ counts the number of bijective functions $f : [n] \to [n]$.

The following estimates hold for n large.

Proposition 1.1. *For $n \geq 1$,*

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n.$$

The following lemma is helpful.

Lemma 1.2. *For every real number x,*

$$1 + x \leq e^x.$$

Exercise 1.3.

1. *Prove Lemma 1.2.*
2. *Show that the number e is characterized by Lemma 1.2. That is, show that if there exists a number a such that*

$$1 + x \leq a^x$$

for every x, then $a = e$.

Exercise 1.4. *Prove Proposition 1.1. (Hint: induction on n.)*

(This proof may be found in Chapter 3 of [1].)

When n is large, an even better approximation is given by Stirling's formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Here we use the notation $f(n) \approx g(n)$ to mean that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

1.2 Binomial coefficients

Define the binomial coefficient $\binom{n}{i}$ by

$$\binom{n}{k} = \frac{n!}{k!}(n-k)!.$$

It is hard to overestimate the usefulness of the binomial theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i},$$

This definition of binomial coefficient makes sense for integers $n \geq k \geq 0$. It is sometimes nice to extend this definition to more general settings. If r is an arbitrary real or complex number, and $k \geq 0$ is an integer, define the falling factorial $(r)_k$ by

$$(r)_k = r(r-1)\dots(r-k+1).$$

Then we can define

$$\binom{r}{k} = \frac{(r)_k}{k!}.$$

In particular $\binom{n}{k} = 0$ whenever $k \geq n$.

This more general definition makes it possible to write Newton's generalization of the binomial theorem in a nice way.

Theorem 1.5 (Newton)**.** *For a complex number x with $|x| < 1$ and arbitrary real number α, we have*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Exercise 1.6. *Derive the usual binomial theorem from Newton's generlization.*

From the usual binomial theorem, some useful identities can easily be deduced. For example,

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n,$$

and

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \text{ for } n \geq 1.$$

Exercise 1.7. *Deduce both of these identities without resorting to the binomial theorem, using a "bijective" proof.*

1.3 Multinomial coefficients

A more general version of the binomial theorem is the multinomial theorem. For three variables, for example.

$$(x+y+z)^n = \sum_{a+b+c=n} \frac{n!}{a!b!c!} x^a y^b z^c,$$

Exercise 1.8. *Formulate and prove a k-variable multinomial theorem.*

Note that multinomial coefficients naturally arise in certain types of counting problems. For example, one easily checks that the number of anagrams of the word "bananacabana" is

$$\frac{12!}{6!2!1!3!},$$

where 6, 2, 1, 3 count the number of a's, b's, c's, and n's, respectively. Usually anagrams are required to themselves be recognizable words or sentences but we do not require that here—after all, "bananacabana" is arguably not a "word". So an anagram is simply a permutation of letters.

1.4 Stars and bars

Suppose that one wants to buy n pieces of candy, and there are k different types of candy. It is assumed that two pieces of the same type are indistinguishable, and that there are an unlimited number of each type.

Exercise 1.9. *Show that there are*

$$\binom{n+k-1}{n}$$

different ways to do this.

Exercise 1.10. *Now suppose that you must get at least one piece of each of type. Show that there are*

$$\binom{n-1}{n-k} = \binom{n-1}{k-1}$$

ways to do this.

1.5 Partitions of sets and Stirling numbers of the second kind

Define $[n] = \{1, 2, \ldots, n\}$. The Stirling numbers (of the second kind), denoted $S(n, k)$ or $\left\{ {n \atop k} \right\}$, are the number of partitions of $[n]$ into k nonempty parts.

For example $S(3, 2) = 3$, because

$$\{1, 2, 3\} = \{1, 2\} \cup \{3\} = \{1, 3\} \cup \{2\} = \{1\} \cup \{2, 3\},$$

and there are no other partitions of $[3]$ into two nonempty parts.

Exercise 1.11. *Find formulas for $S(n, 2)$, $S(n, 3)$, and $S(n, 4)$.*

Exercise 1.12. *Find a general formula for $S(n, n-1)$.*

The notation $\left\{ {n \atop k} \right\}$ suggests an analogy to binomial coefficients $\binom{n}{k}$. Similar to the recursion for binomial coefficieints

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

we have the recursion

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}.$$

Exercise 1.13. *Prove this recursion formula.*

Exercise 1.14. *Use the recursion formula and the fact that* $S(n,n) = S(n,1) = 1$ *to compute* $S(n,k)$ *for* $1 \le k \le n \le 5$.

Exercise 1.15. *Prove that* $\left\{ {n \atop n-k} \right\}$ *is a polynomial in* n *for each fixed* k.

Another important identity is the following.

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)\dots(x-k+1) = x^n.$$

Exercise 1.16. *(1) Prove that*

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

In particular, the right-hand side is zero for $k > n$. *(Done in class.)*

(2) Let B_n *be the* n*th Bell number, i.e. the total number of partitions of* $[n]$. *Prove that*

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

1.6 Partitions of numbers

A partition of an integer n is a sum

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

where

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k \ge 1.$$

Remark 1.17. *Please note the difference: a partition of* $[n]$ *is a set partition, as in the previous section. A partition of* n *is a number partition.*

For example, for $n = 4$ a complete list of partitions is as follows.

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

Let $p(n)$ denote the number of partitions of n. By the above, $p(4) = 5$. A *generating function* for $p(n)$ can be given as follows.

$$\begin{aligned} \sum_{i=1}^{\infty} p(i)x^i &= \prod_{i=1}^{\infty} \frac{1}{1-x^i} \\ &= (1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\dots \end{aligned}$$

This can be regarded as a purely formal power series, where all we care about is that the coefficient of x^i is $p(i)$. On the other hand, from the point of view of analysis (and analytic number theory) it is worth noting that this series converges for $x \in (-1, 1)$.

Exercise 1.18. *Write a generating function for the number of ways to make n cents with pennies, nickels, dimes, quarters, and half-dollars.*

To compute the partition function $p(n)$ for small values of n, it is useful to have a recursion. Here is it is helpful to introduce an intermediary function $p(k, n)$, the number of partitions of n where each part has size at least k. (So our original function is $p(n) = p(1, n)$.

Noting that $p(k, n) = 0$ if $k > n$ and $p(k, n) = 1$ if $k = n$, we can compute the rest of the values with the recursion

$$p(k, n) = p(k+1, n) + p(k, n-k).$$

Exercise 1.19. *Why does this recursion hold?*

Exercise 1.20. *Use the recursion formula to compute $p(k, n)$ for $0 \le k \le n \le 10$.*

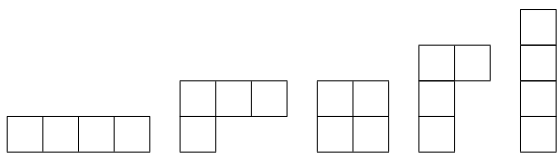

Figure 1: The five partitions of $n = 4$

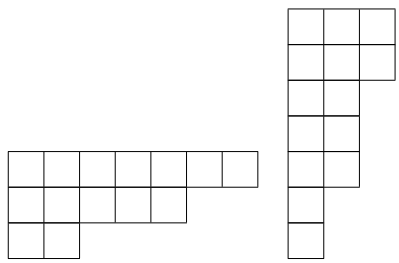

Figure 2: A pair of conjugate partitions of $n = 14$

There are a number of interesting identities involving partitions, and we will discuss a few of the most famous in this section. When discussing partitions, it is sometimes convenient to use Young diagrams, as in Figure 1.

The use of Young diagrams makes the proof of the following almost immediate.

Theorem 1.21. *The number of partitions of n into at most k parts is equal to the number of partitions of n into parts of size at most k,*

Proof. For every partition of n there is a conjugate partition, made by exchanging columns and rows. For example, the partition of $(7, 5, 2)$ is conjugate to the partition $(3, 3, 2, 2, 2, 1, 1)$, as in Figure 2. The result immediate follows. $\square$

Exercise 1.22. *Show that the number of partitions of n into no more than k parts is the same as the number of partitions of $n + k$ into exactly k parts.*

Proposition 1.23. *The number of partitions into distinct parts is the same as the number of partitions into odd parts.*

This is a famous theorem of Euler.

Let $p_k(n)$ denote the number of partitions of n into exactly k parts.

Exercise 1.24. *Show that* $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$.

Exercise 1.25. *Show that the number of partitions of* n *into at most* k *parts is equal to* $p_k(n+k)$.

For a real number x, let $[x]$ denote the nearest integer to x. The following is a nice application of generating functions.

Proposition 1.26.

$$p_3(n) = \left[\frac{n^2}{12}\right].$$

We require the following lemma, which follows from Newton's generalized binomial theorem.

Lemma 1.27.

$$\frac{1}{(1-x)^{a+1}} = \sum_{j=0}^{\infty} \binom{a+j}{j} x^j.$$

Proof of Proposition 1.26. We have that $p_3(n)$ is the number of solutions into exactly 3 parts. It is somewhat easier to work with $a_3(n)$, which we define to be the number of partitions into at most 3 parts. Equivalently, $a_3(n)$ is the number of partitions into parts of size at most 3. Clearly $a_3(n) = p_3(n+3)$.

The generating function for $a_3(n)$ is given by

$$f(x) = \sum_{n=0}^{\infty} a_3(n) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right).$$

A partial fractions decomposition gives that

$$\begin{aligned} f(x) = {} & \frac{1}{6}(1-x)^{-3} + \frac{1}{4}(1-x)^{-2} + \frac{17}{72}(1-x)^{-1} \\ & + \frac{1}{8}(1+x)^{-1} + \frac{1}{9}(1-\omega x)^{-1} + \frac{1}{9}(1-\omega^2 x)^{-1}. \end{aligned}$$

where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity.

Then applying Lemma 1.27 we have that

$$a_3(n) = \frac{1}{12}(n+3)^2 - \frac{7}{72} + \frac{(-1)^n}{8} + \frac{1}{9}\left(\omega^n + \omega^{2n}\right),$$

and therefore

$$\left| a_3(n) - \frac{1}{12}(n+3)^2 \right| \leq \frac{7}{72} + \frac{1}{8} + \frac{2}{9} < \frac{1}{2}.$$

Since $a_3(n) = p_3(n+3)$, we conclude that

$$\left| p_3(n) - \frac{1}{12}n^2 \right| < \frac{1}{2},$$

as desired. $\square$

Proposition 1.26 and its proof were taken from Chapter 15 of van Lint and Wilson's book [5].

1.6.1 More on partitions

The generating function for $p(n)$ is given by

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-x^k} \right).$$

Euler proved the *pentagonal number theorem*, that

$$\begin{aligned} P(x)^{-1} &= (1-x)(1-x^2)(1-x^3)\dots \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \dots \end{aligned}$$

We sketch a proof of this.

$$P(x)^{-1} = \prod_{i=1}^{\infty} (1-x^i)$$

We expand the first few terms of the product, to see if we notice any pattern.

$$\begin{aligned} (1-x)(1-x^2)(1-x^3)(1-x^4)\dots &= 1 - x - x^2 + (x^{1+2} - x^3) + (x^{1+3} - x^4) \\ &\quad + (-x^{1+2+3} + x^{1+5} + x^{2+4} - x^6) + (-x^{4+2+1} + x^{1+6} + x^{2+5} + x^{3} \\ &= 1 - x - x^2 + x^5 + x^7 - \cdots \end{aligned}$$

So we see that $P(x)^{-1}$ is the generating function for the number of partitions of n into an even number of distinct parts minus the number of partitions of n into an odd number of distinct parts. Intuitively it seems this quantity should be small, but surprising much more than that can be said. In fact this quantity is only ever 0, 1, or -1. We'll see this by constructing a partial involution between the two types of partitions.

Given a partition of n into distinct parts let m be the smallest part and let s be the length of the longest run of parts l, $l-1$,..., $l-(s-1)$ where l is the largest part of the partition. Note that we can think of s as the length of the rightmost 45° diagonal in the diagram of the partition.

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & & & & \end{array}$$

Above we see a partition where $m = 1$ and $s = 2$.

Now to define our partial involution. If $s < m$ then we subtract 1 from the s largest parts and add a part of size s. Essentially we remove the rightmost diagonal and make it the new smallest part. If $m \leq s$ then we remove the smallest part (the part of size m) and add 1 to the m largest parts. Clearly, when this function is defined, it is an involution that changes the parity of the number of parts of a partition.

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & & & \end{array} \quad \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \end{array} \quad \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \end{array}$$

Now there are no complications in defining our involution provided the rightmost diagonal and the smallest component don't intersect. Suppose they do and $s < m$. Then we need s to be the smallest part even after we subtract 1 from m. So we need $s < m - 1$. Now suppose the rightmost diagonal and the smallest component intersect and $m \leq s$, then we need there to still be m parts after we remove the smallest part. Note that when the rightmost diagonal and the smallest component intersect then s is the total number of components. So we need $m \leq s + 1$.

So our problem cases are when the rightmost diagonal and the smallest component intersect and $s = m$ or $s = m - 1$. Lets illustrate these cases for

some small values of s.

$s = 1$

$\bullet$

$n = 2$

$\bullet \quad \bullet$ odd number of parts

$n = 2$

$s = 2$

$\bullet \quad \bullet \quad \bullet$

$\bullet \quad \bullet$

$n = 5$

$\bullet \quad \bullet \quad \bullet \quad \bullet$ even number of parts

$\bullet \quad \bullet \quad \bullet$

$n = 7$

$s = 3$

$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$

$\bullet \quad \bullet \quad \bullet \quad \bullet$

$\bullet \quad \bullet \quad \bullet$

$n = 12$

$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$ odd number of parts

$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$

$\bullet \quad \bullet \quad \bullet \quad \bullet$

$n = 15$

So for each s there are two bad cases (unpaired partitions) each with s parts. In one partition $n = s + (s+1) + \cdots + (s + (s-1)) = s(3s-1)/2$ and one with s more parts so $n = s(3s-1)/2 + s = s(3s+1)/2$. These numbers n are the generalized pentagonal numbers. So we have

$$\begin{aligned} P(x)^{-1} &= 1 + \sum_{i=1}^{\infty} \left[(-1)^i \left(x^{i(3i-1)/2} + x^{i(3i+1)/2}\right)\right] \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots \end{aligned}$$

Noting that plugging $-i$ into $i(3i+1)/2$ yields $i(3i-1)/2$ we can rewrite the sum to give us $P(x)^{-1} = \sum (-1)^i x^{i(3i-1)/2}$ where the sum now ranges over all integers.

Now what does this tell us about $p(n)$? For a moment lets write $P(x)^{-1} = \sum a_n x^n$. We know that $P(x)P(x)^{-1} = 1$. On the level of coefficients this gives us a sequence of equations.

$$\begin{aligned} 1 &= a_0 p(0) \\ 0 &= a_0 p(1) + a_1 p(0) \\ 0 &= a_0 p(2) + a_1 p(1) + a_2 p(0) \\ &\vdots \\ 0 &= a_0 p(n) + a_1 p(n-1) + \cdots + a_n p(0) \\ &\vdots \end{aligned}$$

Now we can solve for $p(n)$ in terms of the smaller partition numbers. We also know that most of the a_i's are 0 and exactly what there are when then are nonzero. This gives the recursive formula

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots$$

or more precisely for $g_i = i(3i-1)/2$

$$p(n) = \sum_{i \neq 0} (-1)^i p(n - g_i)$$

Note this sum is finite since $p(n) = 0$ for $n < 0$.

The numbers appearing as exponents are the (generalized) pentagonal numbers $g_k = k(3k-1)/2$ for $k = 0, \pm 1, \pm 2, \ldots$. particular n.

1.6.2 Congruences and asymptotics

Ramanujan proved a number of congruence formulas for partitions. For example, he showed that for every $k \geq 0$,

$$\begin{aligned} p(5k+4) &\equiv 0 \pmod{5}, \\ p(7k+5) &\equiv 0 \pmod{7}, \\ \text{and} \quad p(11k+6) &\equiv 0 \pmod{11}. \end{aligned}$$

An asymptotic formula for $p(n)$ due to Hardy and Ramanujan is

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}},$$

as $n \to \infty$.

1.7 The Twelvefold Way

> "I will shamelessly tell you what my bottom line is. It is placing balls into boxes, or, as Florence Nightengale David put it with exquisite tact in her book *Combinatorial Chance*, it is the theory of distribution and occupancy."
> — Gian-Carlo Rota, Indiscrete Thoughts [2]

We will describe balls as the set $[n]$ and boxes as the set $[k]$, and putting the balls into boxes as a function $f : [n] \to [k]$.

Restricting to injective, surjective, or arbitrary f gives three possibilities. Whether balls are distinguishable gives two, and whether boxes are distinguishable gives two more. We could be more precise about what "not distinguishable" means. Formally, if the balls are indistinguishable we are counting orbits of the natural action of the symmetric group $\mathcal{S}_n$ on the set of functions $f : [n]\ to[k]$.

There are a total of twelve possibilities, and these possibilities are what we call the Twelvefold Way. According to Stanley, this classification was suggested by Rota and the name was suggested by Joel Spencer [3].

1.7.1 f without restriction

1. Suppose first that balls and boxes are both distinguishable — then by the multiplicative principle there are

$$k^n$$

choices.

2. Suppose that balls are not distinguishable, but boxes are — then we are back to "stars and bars" of Section 1.4, and there are

$$\binom{n+k-1}{n}$$

possibilities. (We only have to count how many balls land in each box...)

3. Suppose that balls are distinguishable, and boxes are not — then this is a set partition, and there are

$$\sum_{i=1}^{k} \left\{ {n \atop i} \right\}$$

ways to do it.

4. Finally, suppose that neither balls nor boxes are distinguishable — then this is a number partition and there are

$$\sum_{i=1}^{k} p_i(n)$$

ways to put the balls in boxes. Note that by Exercise 1.25 this sum can be rewritten as

$$p_k(n+k).$$

1.7.2 Injective f

1. Balls and boxes both distinguishable. There are

$$(k)_n = k(k-1)\dots(k-n+1)$$

ways to do this, by the multiplicative principle.

2. Balls not distinguishable, but boxes are. Then there are

$$\binom{k}{n}$$

ways. Choose which boxes will contain a ball.

3. Now if boxes are not distinguishable (whether or not balls are), the case of injective maps becomes simple. There is 1 way if $n \leq k$ and there are 0 ways otherwise.

1.7.3 Surjective f

1. Balls and boxes both distinguishable. Think of this as a set partition, and then permuting the sets. So there are

$$k! \left\{ {n \atop k} \right\}.$$

2. Balls not distinguishable, but boxes are. This stars and bars, where you have to have at least one of each type. So there are

$$\binom{n-1}{k-1}$$

ways total.

3. Balls are distinguishable, but boxes are not. This is exactly the definition of
$$\left\{ {n \atop k} \right\}.$$

4. Neither balls nor boxes are distinguishable. This is the definition of
$$p_k(n).$$

1.7.4 Summary

The Twelvefold Way is summarized in the following table. (Adapted from the table on p. 14 of [3].)

n **balls**	k **boxes**	**Arbitrary** f	**Injective** f	**Surjective** f
distinguishable	distinguishable	k^n	$(k)_n$	$k! \left\{ {n \atop k} \right\}$
indistinguishable	distinguishable	$\binom{n+k-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{k-1}$
distinguishable	indistinguishable	$\sum_{i=1}^{k} \left\{ {n \atop i} \right\}$	1 or 0	$\left\{ {n \atop k} \right\}$
indistinguishable	indistinguishable	$\sum_{i=1}^{k} p_i(n)$	1 or 0	$p_k(n)$

1.8 Catalan numbers

Catalan numbers were probably first studied by Euler, and even earlier by the Mongolian astronomer and mathematician Minggantu.

There are a host of things which Catalan numbers C_n count. See, for example, the 66 combinatorial interpretations for C_n in exercise 6.19 of Stanley's EC2 [4], which invites the reader to give $\binom{66}{2} = 2145$ bijections between pairs of these interpretations. As a few representative examples, the Catalan number C_n counts:

1. Dyck words of length $2n$ (definition below).

2. Monotone lattice paths from $(0, 0)$ to (n, n) which never pass below the diagonal line $y = x$.

3. Triangulations of the $(n + 2)$-gon.

4. The number of ways of associating n applications of a binary operator to $n+1$ factors.

5. Full binary trees with $n+1$ leaves.

6. Stack-sortable permutations on $[n]$, i.e. permutations which avoid the pattern 231.

Definition 1.28. *A Dyck word of length $2n$ has n X's and n Y's, and as you read the word left-to-right, the number of X's is always at least the number of Y's.*

For example, if $n = 3$ there are 5 Dyck words of length $2n$.

$$\begin{gathered} XXXYYY \\ XXYXYY \\ XXYYXY \\ XYXXYY \\ XYXYXY \end{gathered}$$

Exercise 1.29. *Check that Dyck words satisfy the recurrence $C(0) = 1$.*

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Exercise 1.30. *Check that the number of triangulations of the $(n+2)$-gon satisfies the same recurrence.*

If we define the generating function

$$c = c(x) = \sum_{i=0}^{\infty} C_i x^i$$

then the recurrence relation immediately gives the formula

$$xc^2 - c + 1 = 0 \tag{1}$$

Then one can derive the formula

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Exercise 1.31. *The degree-two polynomial in equation (1) has two solutions. Why do we get the above formula, and not*

$$c(x) = \frac{1 + \sqrt{1 - 4x}}{2x}?$$

Hint: consider the behavior of $c(x)$ near $x = 0$.

From Newton's general form of the binomial theorem, we have that

$$\sqrt{1+y} = \sum_{i=0}^{\infty} \binom{1/2}{n} y^n.$$

After a little algebra, we have that

$$\sqrt{1+y} = \sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \left(\frac{1}{1-2n}\right) \binom{2n}{n} y^i.$$

Substituting $y = -4x$, we derive the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The formula may also be derived combinatorially, without resorting to generating functions. The following lemma may be proved by a "reflection trick".

Lemma 1.32. *The number of monotone lattice paths from $(0,0)$ to (n,n) which cross the diagonal $y = x$ are in bijection with the total number of monotone lattice paths from $(0,0)$ to $(n+1, n-1)$.*

We note that the total number of monotone lattice paths from $(0,0)$ to (a,b) is $\binom{a+b}{a}$. This is clear, since once must take $a+b$ steps in some order, a steps to the right and b steps up.

Then if we consider paths crossing the diagonal to be "bad paths" and paths staying under or on the diagonal to be "good paths", the total number of paths is $\binom{2n}{n}$. So we have

$$\begin{aligned} C_n &= \binom{2n}{n} - \binom{2n}{n+1} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

1.9 Counting under symmetry

Many combinatorial situations call for counting under symmetry.

Consider a group G acting on a set X. The following lemma [1] is useful for counting orbits of the G action.

Lemma 1.33 (Burnside's Lemma)**.**

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Here X/G denotes the set of orbits of the group action. We also have the notation for X^g, the set of fixed points of a group element g,

$$X^g = \{x \in X \mid g \cdot x = x\}.$$

An example application is the following.

Count the number of n-colorings of the six faces of a cube, up to rotation. If we somehow number the faces $1 \ldots 6$, we can think of a coloring as a function $f : [6] \to n$. The group of rotations of the cube acts on the set of faces $[6]$, so it also acts on the set of such functions $[n] \to 6$.

If we do not allow for rotation, there are n^6 possibilities, but of course there are fewer if we consider two colorings the same if they only differ by a rotation of the cube.

There are 24 rotations of the cube, and it is convenient to categorize them into a few different types. We count the number of each type.

axis of rotation	**angle**	**number of rotations**
identity	none	1
pair of opposite vertices	$\pm 2\pi/3$	8
pair of opposite edges	π	6
pair of opposite faces	$\pm\pi/2$	6
pair of opposite faces	π	3

Now Lemma 1.33 tells us that to count the number of orbits, we just have to count the number of elements fixed by each group element.

How many of the n^6 functions $[6] \to n$ are fixed by the identity element? Plainly all n^6.

[1] It is called Burnside's Lemma, even though Burnside attributed it to Frobenius, who in turn credits Cauchy.

axis of rotation	number of fixed colorings	number of rotations
identity	n^6	1
pair of opposite vertices	n^2	8
pair of opposite edges	n^3	6
pair of opposite faces	n^3	6
pair of opposite faces	n^4	3

We conclude that the number of rotationally distinct colorings is

$$\begin{aligned} X/G &= \frac{1}{|G|}\sum_{g\in G}|X^g| \\ &= \frac{1}{24}\left(n^6+8n^2+6n^3+6n^3+3n^4\right) \\ &= \frac{1}{24}\left(n^6+3n^4+12n^3+8n^2\right). \end{aligned}$$

We can double check our answer at the Online Encyclopedia of Integer Sequences (OEIS). This formula is given as sequence number A047780 "Number of inequivalent ways to color faces of a cube using at most n colors."

Exercise 1.34. *If reflections of the cube are allowed, as well as rotations, then there are* 48 *symmetries in total. How many ways are there to n-color the faces of the cube, if two colorings are considered the same whenever they are equivalent up to rotation or reflection?*

Exercise 1.35. *How many ways are there to n-color the* edges *of a tetrahedron, up to rotation?*

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