Math 4575 : HW #3

Chapter 5: #6, 12, 14, 15, 18, 19, 20, 24, 29, 39, 47

- #6
  What is the coefficient of x^5y^{13} in the expansion of $(3x-2y)^{18}$? What is the coefficient of x^8y^9? (There is not a misprint in this last question!)
  The coefficient of x^5y^{13} is given by the binomial theorem: $\binom{18}{5}3^5(-2)^{13}$.
  The coefficient of x^8y^9 is zero, since $(3x-2y)^{18}$ is a polynomial which is homogeneous of degree 18.

- #12
  Let n be a positive integer. Prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m\binom{2m}{m} & \text{if } n \text{ is even} \end{cases}$$

We consider two cases, n odd and n even.
First, suppose that n is odd. Then

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 &= \binom{n}{0}^2 - \binom{n}{1}^2 + \cdots + (-1)^n\binom{n}{n}^2 \\
&= \left(\binom{n}{0}^2 - \binom{n}{n}^2\right) + \left(\binom{n}{1}^2 - \binom{n}{n-1}^2\right) + \left(\binom{n}{2}^2 - \binom{n}{n-2}^2\right) + \dots \\
&= \left(\binom{n}{0}^2 - \binom{n}{0}^2\right) + \left(\binom{n}{1}^2 - \binom{n}{1}^2\right) + \left(\binom{n}{2}^2 - \binom{n}{2}^2\right) + \dots \\
&= 0
\end{aligned}$$

Next, suppose that n is even, i.e. let $n = 2m$ for some m. Then we can write

$$(*) \qquad (1-x^2)^n = (1-x)^n(1+x)^n,$$

and consider the coefficient of x^n on both sides.
Expanding the left side of $(*)$ via the binomial theorem, we obtain

$$(1-x^2)^n = \sum_{k=0}^{n}\binom{n}{k}x^{2k},$$

so the coefficient of x^n is

$$(-1)^m\binom{2m}{m}.$$

Expanding the right side of $(*)$, we have

$$(1-x)^n(1+x)^n = \left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}\right)\left(\sum_{j=0}^{n}\binom{n}{j}\right)$$

Multiplying these two polynomials, to find the coefficient of x^n we sum

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}^2,$$

since $\binom{n}{k} = \binom{n}{n-k}$. Since the two polynomials in $(*)$ are equal, their coefficient of x^n must agree. We have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = (-1)^m\binom{2m}{m},$$

as desired.

- #14
  Prove that
  $$\binom{r}{k} = \frac{r}{r-k}\binom{r-1}{k}$$
  for r a real number and k an integer with $r \neq k$.
  We have
  $$\begin{aligned}\frac{r}{r-k}\binom{r-1}{k} &= \frac{r}{r-k} \times \frac{(r-1)(r-2)\dots(r-k)}{k!}\\ &= \frac{r(r-1)\dots(r-k+1)}{k!}\\ &= \binom{r}{k}.\end{aligned}$$

  Comment: The equality only holds when $r \neq k$, since we have to be careful about division by zero.

- #15
  Prove, that for every integer $n > 1$,
  $$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \dots + (-1)^{n-1}n\binom{n}{n} = 0.$$
  Using the binomial theorem, we expand:
  $$(1-x)^n = 1 - \binom{n}{1}x^1 + \binom{n}{2}x^2 - \dots + (-1)^n\binom{n}{n}x^n.$$
  We differentiate both sides with respect to x to obtain:
  $$-n(1-x)^{n-1} = -\binom{n}{1} + 2\binom{n}{2}x^1 - \dots + (-1)^n n\binom{n}{n}x^{n-1}.$$
  Multiplying both sides by -1 and then evaluating both sides at $x = 1$ gives the desired result.

- #18
  Evaluate the sum
  $$1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\frac{1}{4}\binom{n}{3}+\cdots+(-1)^n\frac{1}{n+1}\binom{n}{n}.$$
  Start with
  $$(1+x)^n=1+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n.$$
  Integrate both sides to obtain
  $$\frac{(1+x)^{n+1}}{n+1}=C+x+\frac{1}{2}\binom{n}{1}x^2+\cdots+\frac{1}{n+1}\binom{n}{n}x^{n+1},$$
  where C is a constant of integration. Evaluate both sides at $x=0$ to find that
  $$C=\frac{1}{n+1}.$$
  So now we have
  $$\frac{(1+x)^{n+1}}{n+1}=\frac{1}{n+1}+x+\frac{1}{2}\binom{n}{1}x^2+\cdots+\frac{1}{n+1}\binom{n}{n}x^{n+1}.$$
  Evaluate both sides at $x=-1$ to find that
  $$0=\frac{1}{n+1}-1+\frac{1}{2}\binom{n}{1}+\cdots+(-1)^{n+1}\frac{1}{n+1}\binom{n}{n}.$$
  Then
  $$1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\frac{1}{4}\binom{n}{3}+\cdots+(-1)^n\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}.$$

- #19
  Sum the series $1^2+2^2+3^2+\cdots+n^2$ by observing that
  $$m^2=2\binom{m}{2}+\binom{m}{1}$$
  and using the identity (5.19), that
  $$\binom{n+1}{k+1}=\binom{0}{k}+\binom{1}{k}+\cdots+\binom{n}{k}.$$
  First we check that, indeed,
  $$\begin{aligned}2\binom{m}{2}+\binom{m}{1}&=m(m-1)+m\\&=m^2.\end{aligned}$$

Now, we write

$$\begin{aligned}\sum_{m=1}^{n} m^2 &= \sum_{m=1}^{n}\left(2\binom{m}{2}+\binom{m}{1}\right)\\ &= 2\sum_{m=1}^{n}\binom{m}{2}+\sum_{m=1}^{n}\binom{m}{1}\\ &= 2\binom{n+1}{3}+\binom{n+1}{2}.\end{aligned}$$

If you like, this can be rewritten as

$$\frac{(2n+1)(n+1)(n-1)}{6}.$$

- #20
  Find integers a, b, and c such that

$$m^3 = a\binom{m}{3}+b\binom{m}{2}+c\binom{m}{1}$$

for all m. Then sum the series $1^3+2^3+3^3+\cdots+n^3$.
First, we solve for a, b, and c. Both sides of the equation are polynomials of degree 3. By considering the coefficient of m^3 on both sides, we obtain $a=6$. Evaluating both sides at $m=1$ we obtain $c=1$, and then evaluating both sides at $m=2$ we obtain $b=6$. So

$$m^3 = 6\binom{m}{3}+6\binom{m}{2}+\binom{m}{1}.$$

Now, we have

$$\begin{aligned}\sum_{m=1}^{n} m^3 &= \sum_{m=1}^{n}\left(6\binom{m}{3}+6\binom{m}{2}+\binom{m}{1}\right)\\ &= 6\sum_{m=1}^{n}\binom{m}{3}+6\sum_{m=1}^{n}\binom{m}{2}+\sum_{m=1}^{n}\binom{m}{1}\\ &= 6\binom{n+1}{4}+6\binom{n+1}{3}+\binom{n+1}{2}.\end{aligned}$$

This can be rewritten as

$$\frac{(n+1)^2n^2}{4}.$$

Comment: an equivalent formulation is the following attractive identity.

$$1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2.$$

See more on various proofs of this at the Wikipedia page
`https://en.wikipedia.org/wiki/Squared_triangular_number`

- #24
  Consider a three-dimensional grid whose dimensions are 10 by 15 by 20. You are at the front lower left corner of the grid and wish to get to the back upper right corner 45 "blocks" away. How many different routes are there in which you walk exactly 45 blocks?
  One takes 45 steps, 10 steps to the right, 15 steps up, and 20 steps forward. This is the same as counting sequences of 45 letters where 10 are R, 15 are U, and 20 are F. This is counted by the multinomial coefficient

$$\binom{45}{10, 15, 20} = \frac{45!}{10!15!20!}.$$

- #29
  Find and prove a formula for

$$\sum \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t},$$

  where the summation extends over all nonnegative integers r, s, and t with sum $r + s + t = n$.
  We claim that

$$\sum \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t} = \binom{m_1 + m_2 + m_3}{n}.$$

  Suppose that we have m_1 Democrats, m_2 Independents, and m_3 Republicans, and we want to make a committee with n people total. Both sides count the number of ways to do this: we could just put everyone together and then choose the committee of n—there are $\binom{m_1+m_2+m_3}{n}$ ways to do this. On the other hand, there are $\binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}$ ways to choose a committee with r Democrats, s Independents, and t Republicans. Summing this over all nonnegative solutions to $r + s + t = n$ gives all the possibilities.

- #39
  Determine the coefficient of $x_1^3 x_2 x_3^4 x_5^2$ in the expansion of

$$(x_1 + x_2 + x_3 + x_4 + x_5)^{10}.$$

  By the multinomial theorem, this coefficient is

$$\binom{10}{3, 1, 4, 2} = \frac{10!}{3!1!4!2!}.$$

- #47
  Use Newton's binomial theorem to approximate $10^{1/3}$.
  First, note that

$$\begin{aligned} 10^{1/3} &= (8+2)^{1/3} \\ &= 2\left(1+\frac{1}{4}\right)^{1/3} \end{aligned}$$

  Newton's binomial theorem gives that

$$\left(1+\frac{1}{4}\right)^{1/3} = \sum_{k=0}^{\infty} \binom{1/3}{k} \left(\frac{1}{4}\right)^{1/k}.$$

  Taking the first few terms,

$$10^{1/3} \approx 2\left(1+\left(\frac{1}{3}\right)\left(\frac{1}{1!}\right)\left(\frac{1}{4}\right)+\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{1}{2!}\right)\left(\frac{1}{4}\right)^2+\dots\right)$$

  If we sum just these first three terms and reduce the fraction, we obtain the approximation

$$10^{1/3} \approx \frac{155}{72}.$$

  This is not bad, since $10^{1/3} = 2.15443\dots$, and $155/72 = 2.15278\dots$.
  How close an approximation do you get if you include the next term?