Math 4575 : HW #4

Chapter 6: #1, 3, 7, 9, 11

- #1

  Find the number of integers between 1 and 10000 inclusive that are not divisible by 4,5, or 6.

  Let S be the set of all numbers 1 to 10000, Let A_1 be the subset of numbers divisible by 4, A_2 the subset of numbers divisible by 5, and A_3 the subset of numbers divisible by 6. By the inclusion-exclusion principle,

  $$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|.$$

  We have $|S| = 10000$, $|A_1| = 2500$, $|A_2| = 2000$, $|A_3| = \lfloor 10000/6 \rfloor = 1666$. Since $\text{lcm}(4,5) = 20$, $\text{lcm}(4,6) = 12$, and $\text{lcm}(5,6) = 30$, we also have $|A_1 \cap A_2| = 500$, $|A_1 \cap A_3| = 833$, and $|A_2 \cap A_3| = 333$. Finally, since $\text{lcm}(4,5,6) = 60$, we have $|A_1 \cap A_2 \cap A_3| = 166$. Putting it altogether, we have

  $$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= 10000 - 2500 - 2000 - 1666 + 500 + 833 + 333 - 166 \\ &= 5334. \end{aligned}$$

- #3

  Find the number of integers between 1 and 10000 that are neither perfect squares nor perfect cubes.

  Let S be the set of integers between 1 and 10000. Let A_1 be the set of perfect squares, and A_2 the set of perfect cubes. Clearly $A_1 = \sqrt{10000} = 100$, and $|A_2| = \lfloor 10000^{1/3} \rfloor = 21$. A number is both a perfect square and a perfect cube, if and only if it is a perfect sixth power, by the Fundamental Theorem of Arithmetic. So $A_1 \cap A_2 = \lfloor 10000^{1/6} \rfloor = 4$.

  So by the inclusion-exclusion principle, the number of integers in S that are neither perfect squares nor perfect cubes is

  $$10000 - 100 - 21 + 4 = 9833.$$

- #7

  Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in nonnegative integers x_1, x_2, x_3, and x_4 not exceeding 8.

  Let S be the set of nonnegative integer solutions to the equation. By stars and bars counting, $|S| = \binom{17}{3}$. Let A_1 be the number of solutions where $x_1 \geq 9$. Subtracting 9 everywhere, this is the same as the number of nonnegative integer solutions to $y_1 + y_2 + y_3 + y_4 = 5$, or $\binom{8}{3}$. Similarly define A_2, A_3, and A_4. The same calculation shows that $A_i = \binom{8}{3}$ for $i = 1, 2, 3, 4$. But it is not possible for two of the variables to both be 9 or greater, since then the sum is already greater than 14. So the pairwise (and higher) intersections of these sets are all empty. So the number of solutions is

  $$\binom{17}{3} - 4\binom{8}{3}.$$

- #9
  Determine the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

that satisfy

$$1 \le x_1 \le 6,\ 0 \le x_2 \le 7,\ 4 \le x_3 \le 8,\ 2 \le x_4 \le 6.$$

First, make the variable substitutions $y_1 = x_1 - 1$, $y_2 = x_2$, $y_3 = x_3 - 4$, and $y_4 = x_4 - 2$. Then we are counting integral solutions to

$$(*) \quad y_1 + y_2 + y_3 + y_4 = 13,$$

that satisfy

$$0 \le y_1 \le 5,\ 0 \le y_2 \le 7,\ 0 \le y_3 \le 4,\ 0 \le y_4 \le 4.$$

Let S denote the number of nonnegative integer solutions to $(*)$. Let A_1 be the subset of solutions with $y_1 \ge 6$, A_2 the subset of solutions with $y_2 \ge 8$, A_3 the subset of solutions with $y_3 \ge 5$, and A_4 the subset of solutions with $y_4 \ge 5$.
We have $|S| = \binom{16}{3}$, $|A_1| = \binom{10}{3}$, $|A_2| = \binom{8}{3}$, $|A_3| = \binom{11}{3}$, and $|A_4| = \binom{11}{3}$.
Now compute the sizes of the pairwise intersections. We have $|A_1 \cap A_2| = 0$, $|A_1 \cap A_3| = \binom{5}{3}$, $|A_1 \cap A_4| = \binom{5}{3}$, $|A_2 \cap A_3| = \binom{3}{3}$, $|A_2 \cap A_4| = \binom{3}{3}$, and $|A_3 \cap A_4| = \binom{6}{3}$. The intersection of any three (or more) of these subsets is empty. So

$$\begin{aligned}|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| &= |S| - |A_1| - |A_2| - |A_3| - |A_4| + |A_1 \cap A_2| + |A_1 \cap A_3| + \dots \\ &= \binom{16}{3} - \binom{10}{3} - \binom{8}{3} - 2\binom{11}{3} + 2\binom{5}{3} + 2\binom{3}{3} + \binom{6}{3}\end{aligned}$$

- #11
  Determine the number of permutations of $\{1, 2, \dots, 8\}$ in which no even integer is in its natural position.
  By inclusion-exclusion, the number of such permutations is:

$$8! - \binom{4}{1}7! + \binom{4}{2}6! - \binom{4}{3}5! + \binom{4}{4}4!$$