Math 4575 : HW #5

Chapter 6: #12, 14, 21, 29, 32.

- #12
  Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which exactly four integers are in their natural positions.
  First, choose the 4 integers to be in their natural positions, and then choose a derangement of the remaining 4. So the number of such permutations is

$$\binom{8}{4} D_4 = \binom{8}{4} 4! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right).$$

- #14
  Determine a general formula for the number of permutations of the set $\{1, 2, \ldots, n\}$ in which exactly k integers are in their natural positions.
  By the reasoning above, the number of such permutations is

$$\binom{n}{k} D_{n-k}.$$

- #21
  Prove that D_n is an even number if and only if n is an odd number.
  We have the formula

$$\begin{aligned} D_n &= n! \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} \cdots + (-1)^n \frac{1}{n!}\right) \\ &= n! - \frac{n!}{2!} + \cdots - \cdots + (-1)^{n-1} \frac{n!}{(n-1)!} + (-1)^n \frac{n!}{n!} \end{aligned}$$

  Every term in the sum is even, except possibly the last two, since $n(n-1)$ divides all the other terms, and since either n or $n-1$ is even. So we have

$$D_n \equiv (-1)^{n-1} n + (-1)^n \pmod 2.$$

  Since

$$1 \equiv (-1)^{n-1} \pmod 2,$$

  we have

$$D_n \equiv n - 1 \pmod 2,$$

  as desired.

- #29
  A subway has six stops on its route from its base location. There are 10 people on the subway as it departs its base location. Each person exits the subway at one of its six stops, and at each stop at least one person exits. In how many ways

can this happen?
Let S be the set of ways people can get off the subway, if there are no restrictions other than everyone gets off somewhere. Then by the multiplicative principle, $|S| = 6^{10}$.
Now, let A_i be the subset of ways where no one gets off at stop i, for $i = 1, \ldots 6$. By the same reasoning, $|A_i| = 5^{10}$.
All the pairwise intersections have the same size, i.e. $|A_i \cap A_j| = 4^{10}$ and there are $\binom{6}{2}$ such intersections. And so on.
By the inclusion-exclusion principle, the number of ways where at least one person exits at each stop is

$$6^{10} - \binom{6}{1}5^{10} + \binom{6}{2}4^{10} - \binom{6}{3}3^{10} + \binom{6}{4}2^{10} - \binom{6}{5}1^{10} + \binom{6}{6}0^{10}.$$

- #32
Let n be a positive integer and let $p_1, p_2, \ldots, p_k$ be all the distinct primes numbers that divide n. Consider the "Euler function" ϕ defined by

$$\phi(n) = |\{i : 1 \leq i \leq n, \gcd(i, n) = 1\}|.$$

Use the inclusion-exclusion principle to show that

$$\phi(n) = n \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right).$$

Let S be the set of integers $\{1, 2, \ldots, n\}$. For every $i = 1, 2, \ldots, k$, let A_i be the subset of integers m in S such that $p_i \mid m$. We note that

$$|A_i| = \frac{n}{p_i},$$

$$|A_i \cap A_j| = \frac{n}{p_i p_j},$$

etc. By the Fundamental Theorem of Algebra, a number m is relatively prime to n if and only if $p_i \nmid m$ for every $i = 1, \ldots, k$, so

$$\phi(n) = |\bar{A}_1 \cap \bar{A}_2 \cap \ldots \bar{A}_k|.$$

By the inclusion-exclusion principle,

$$\phi(n) = n\left(1 - \sum \frac{1}{p_a} + \sum \frac{1}{p_a p_b} - \sum \frac{1}{p_a p_b p_c} + \ldots\right)$$

On the right side of this equation, the first sum is over all $a = 1, \ldots k$, the second sum is over all pairs, the third sum over all triples, etc. Finally, we apply the identity

$$\prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right) = 1 - \sum \frac{1}{p_a} + \sum \frac{1}{p_a p_b} - \sum \frac{1}{p_a p_b p_c} + \ldots,$$

to conclude that

$$\phi(n) = n\left(1 - \frac{1}{p_j}\right) = 1 - \sum \frac{1}{p_a} + \sum \frac{1}{p_a p_b} - \sum \frac{1}{p_a p_b p_c} + \ldots,$$

as desired.