Math 4575 : HW #7

Chapter 7: #9, 13, 15, 27, 48abc
Chapter 8: #1, 2, 11, 12, 15, 16, 36

Chapter 7

- #9
  Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.

  We see that $h_n = 2h_{n-1} + 2h_{n-2}$ for $n \geq 2$, as follows. If one has a coloring of the chessboard where the rightmost square is red, then the square just to its left is white or blue. Deleting these gives a coloring of a $1 \times (n-2)$ chessboard. If the rightmost square is white or blue, delete and obtain a coloring of a $1 \times (n-1)$ chessboard. This is reversible: taking any coloring of a $1 \times (n-2)$ chessboard, one can extend it by adding either WR or BR on the right. One can also take any coloring of a $1 \times n-1$ chessboard, and add W or B to the right.

  The initial values are $h_0 = 1$, $h_1 = 3$. Our standard techniques for solving linear homogeneous recurrences with constant coefficients gives

$$h_n = \frac{3-2\sqrt{3}}{6}\left(1-\sqrt{3}\right)^n + \frac{3+2\sqrt{3}}{6}\left(1+\sqrt{3}\right)^n.$$

- #13
  Determine the generating function for each of the following sequences:

  (a)
$$1, c, c^2, \ldots, c^n, \ldots$$

$$g(x) = 1 + cx + c^2x^2 + \cdots = \frac{1}{1-cx}.$$

  (b)
$$1, -1, 1, -1 \ldots, (-1)^n, \ldots$$

$$g(x) = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$$

(c)

$$\binom{\alpha}{0}, -\binom{\alpha}{1}, \binom{\alpha}{2}, \ldots, (-1)^n\binom{\alpha}{n}, \ldots \text{ (α is a real number.)}$$

$$g(x) = \binom{\alpha}{0} - \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 - \cdots = (1-x)^\alpha,$$

by Newton's binomial theorem.

(d)

$$1, \frac{1}{1!}, \frac{1}{2!}, \ldots, \frac{1}{n!}, \ldots$$

$$g(x) = 1 + x + \frac{1}{2!}x^2 + \cdots = e^x$$

(e)

$$1, -\frac{1}{1!}, \frac{1}{2!}, \ldots, (-1)^n\frac{1}{n!}, \ldots$$

$$g(x) = 1 - x + \frac{1}{2!}x^2 - \cdots = e^{-x}$$

- #15
  Determine the generating function for the sequence of cubes

$$0, 1, 8, \ldots, n^3, \ldots.$$

Write the generating function for the sequence of cubes as

$$g_3(x) = 0 + x + 8x^2 + 27x^3 + \ldots$$

Next let

$$g_2(x) = 0 + x + 4x^2 + 9x^3 + \ldots$$

be the generating function for squares, and

$$g_1(x) = 0 + x + 2x^2 + 3x^3 + \ldots$$

We notice that

$$x\frac{d}{dx}[g_1] = g_2$$

and

$$x\frac{d}{dx}[g_2] = g_3.$$

So putting these two equations together, we have

$$\begin{aligned} g_3 &= x\frac{d}{dx}\left[x\frac{d}{dx}[g_1]\right] \\ &= x\left(\frac{d}{dx}[g_1] + x\frac{d^2}{dx^2}[g_1]\right) \end{aligned}$$

Since

$$g_1(x) = \frac{x}{(1-x)^2},$$

a little calculus gives the final answer:

$$g_3(x) = \frac{x^3+4x^2+x}{(x-1)^4}.$$

- #27
  Determine the number of n-digit numbers with all digits odd, such that 1 and 3 each occur a nonzero, even number of times.

  This problem is easiest with exponential generating functions. If h_n is the number of allowable n-digit numbers, we define the exponential generating function by

$$g(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}.$$

  We have the following product formula:

$$g(x) = \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right)^2 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^3.$$

  The first factor represents the number of 1's and 3's, which must be even and nonzero. The second factor represents the number of 5's, 7's, and 9's. Now, we have

$$\begin{aligned} g(x) &= \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right)^2 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^3 \\ &= \left(\frac{1}{2}\left(e^x + e^{-x}\right) - 1\right)^2 e^{3x} \\ &= \left(\frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \left(e^x + e^{-x}\right) + 1\right) e^{3x} \\ &= \left(\frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} - e^x - e^{-x} + \frac{3}{2}\right) e^{3x} \\ &= \frac{1}{4}\left(e^{5x} - 4e^{4x} + 6e^{3x} - 4e^{2x} + e^x\right), \end{aligned}$$

so

$$h_n = \frac{1}{4}\left(5^n - 4 \times 4^n + 6 \times 3^n - 4 \times 2^n + 1\right).$$

- #48
  Solve the following recurrence relations by using the method of generating functions as described in Section 7.4:

  (a) $h_n = 4h_{n-2}$, $(n \geq 2)$; $h_0 = 0, h_1 = 1$

  For this one, we don't even need to use the standard methods. Just by observation, $h_n = 0$ is n is even, and $h_n = 4^{n-1}$ if n is odd. This follows immediately from strong induction.

  (b) $h_n = h_{n-1} + h_{n-2}$, $(n \geq 2)$; $h_0 = 1, h_1 = 3$

  This is the same recurrence as Fibonacci. Solving for the initial values gives:

  $$h_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

  (c) $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $(n \geq 3)$; $h_0 = 0, h_1 = 1, h_2 = 2$

  $$h_n = \frac{1}{3}3^n - \frac{1}{12}(-3)^n - \frac{1}{4}.$$

Chapter 8

- #1
  Let $2n$ (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number C_n.

  Let p_n denote the number of ways to join the points in pairs. We have $p_0 = p_1 = 1$. We claim that this sequence satisfies the recursion

  $$p_{n+1} = \sum_{k=0}^{n} p_k p_{n-k}.$$

Indeed, take $2(n+1)$ points on a circle. Fix one of the points P, arbitrarily. In every pairing and joining by line segements, P is connected to some other vertex Q. Let m denote the number of between P to Q, going around the circle clockwise. Then there are $2n-m$ points between P and Q, going around the circle counterclockwise. (The total number of points is $2n+2$, but this includes P and Q.)

If m is odd, there is no way to pair the odd number of points on this side of the circle, so it must be that m (and hence $2n-m$) is even. Write $m = 2k$. There are p_k ways to connect the points on one side of segment PQ and p_{n-k} ways to connect the points on the other side, and these can be done independently of each other. So the multiplicative principle applies, and there are $p_k p_{n-k}$ pairings with P connected to this point Q. Letting Q vary over all choices with m even is the same as letting k range from 0 to n.

Since p_n and C_n satisfy the same recurrence relation and the same initial conditions, they must be the same sequence.

- #2
  Prove that the number of 2-by-n arrays

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \end{bmatrix}$$

that can be made from the numbers $1, 2, \dots, 2n$ such that

$$x_{11} < x_{12} < \cdots < x_{1n},$$

$$x_{21} < x_{22} < \cdots < x_{2n},$$

and

$$x_{11} < x_{21},\ x_{12} < x_{22},\ \dots\ , x_{1n} < x_{2n}$$

equals the nth Catalan number C_n.

We recall that C_n counts the number of Dyck words of length $2n$. Given a Dyck word of length $2n$, let $x_{11} < x_{12} < \cdots < x_{1n}$ encode the positions for letter X, and $x_{21} < x_{22} < \cdots < x_{2n}$ encode the positions for letter Y. For example, if we have the Dyck word $XXYYXYXY$, we encode it as the array

$$\begin{bmatrix} 1 & 2 & 5 & 7 \\ 3 & 4 & 6 & 8 \end{bmatrix}.$$

By construction it satisfies the row inequalities. It satisfies the column inequalities, since it is a Dyck word and the ith Y must came after the ith X. Similarly, we can take any such array and find a Dyck word that it encodes. So this gives a bijection between Dyck words and allowable arrays. The result follows.

- #11
  Compute the Stirling numbers of the second kind $S(8,k)$, $(k = 0, 1, ..., 8)$.

Using the recursion formula, one row at a time, one finds:

$$0, 1, 127, 966, 1701, 1050, 266, 28, 1$$

- #12
  Prove that the Stirling numbers of the second kind satisfy the following relations:

  (a) $S(n, 1) = 1, \ (n \geq 1)$

  This is immediate, from the definition of Stirling numbers.

  (b) $S(n, 2) = 2^{n-1} - 1, \ (n \geq 2)$

  There are $2^n - 2$ ways to choose a nontrivial subset of $[n]$ (not the empty set and not the whole set). If this is one part in the partition, then its complement is the other part, and we have counted each possibility twice.

  (c) $S(n, n-1) = \binom{n}{2}, \ (n \geq 1)$

  Each part must have one element, except one part with two elements. Choose this special part $\binom{n}{2}$ ways.

  (d) $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}, \ (n \geq 2)$

  One possibility is that all the parts have one element, except one part which has 3 elements. Here there are $\binom{n}{3}$ possibilities. The only other possibility is that all the parts have one element, except two parts which each have two elements. Choose the 4 special elements, and then there are $\binom{4}{2}/2 = 3$ ways to pair them into 2 pairs, so $3\binom{n}{4}$ more possibilities.

- #15
  The number of partitions of a set of n elements into k distinguishable boxes (some of which may be empty) is k^n. By counting in a different way, prove that

$$k^n = \binom{k}{1} 1! S(n, 1) + \binom{k}{2} 2! S(n, 2) + \cdots + \binom{k}{n} n! S(n, n).$$

  (If $k > n$, define $S(n, k)$ to be 0.)

  Consider the set of functions $[n] \to [k]$. The left side k^n counts the number of these functions. Now we sort these functions by the size of their image (or

range). If the range is size i then there are $\binom{k}{i} i! S(n,i)$ possible functions: choose the i elements of $[k]$ to be in the image. Partition $[n]$ into i nonempty subsets, and then there are $i!$ ways to map the nonempty subsets to the distinguished elements of $[k]$.

- #16
  Compute the Bell number B_8. (Cf. Exercise 11.)

$$0 + 1 + 127 + 966 + 1701 + 1050 + 266 + 28 + 1 = 4140$$

- #36
  Prove that the Catalan number C_n equals the number of lattice paths from $(0,0)$ to $(2n,0)$ using only "upsteps" $(1,1)$ and "downsteps" $(1,-1)$ that never go above the horizontal axis (so there are as many upsteps as there are downsteps). (These are sometimes called Dyck paths.)

  By rotating such a path by 45 degrees counterclockwise about the origin and then rescaling by a factor of $1/\sqrt{2}$, we obtain a lattice path from $(0,0)$ to (n,n) that never goes above the line $y = x$. This transformation can be inverted (scale by a factor of $\sqrt{2}$ and rotate clockwise 45 degrees), so this gives a bijection between the two kinds of paths.