Math 4575 : HW #7

Chapter 11: #10, 20, 39, 44, 56, 62

- #10
  Prove that any two connected graphs of order n with degree sequence $(2, 2, \ldots, 2)$ are isomorphic.
  We know that a connected graph of order n with this degree sequence admits a closed Eulerian trail passing through every edge. Since it passes through every edge, it passes through every vertex. Because every degree is 2, the trail must enter and exit every vertex exactly once along the trail. In other words, we may label the vertices $v_1, v_2, \ldots, v_n$ such that every vertex v_i is adjacent to v_{i-1} and v_{i+1}. (The indices are read modulo n.) So such a graph must be isomorphic to the cycle C_n.

- #20
  Prove that a graph of order n with at least

  $$\frac{(n-1)(n-2)}{2} + 1$$

  edges must be connected. Give an example of a disconnected graph of order n with one fewer edge.

  If $G = (V, E)$ is disconnected, then it has at least two connected components, and it's vertices may be partitioned $V = A \coprod B$ so that A and B are nonempty, and there are no edges between A and B. Suppose that $A = k$ and $B = n - k$. Then the maximum number of edges in such a graph is

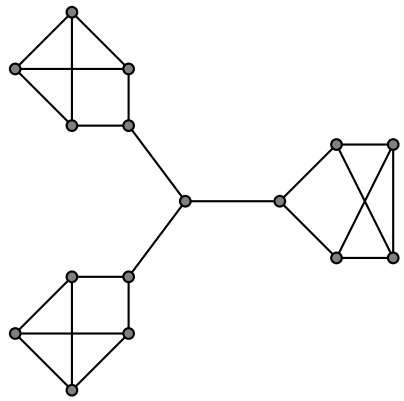
  $$\binom{n}{2} - k(n-k).$$

  Since n is given, maximizing this function is equivalent to minimizing $k(n-k)$. It is straightforward to check that the maximum of the function $\binom{f(k)=(n}{2)-k(n-k)}$ over the interval $k \in [1, n-1]$ are obtained at the endpoints, when $k = 1$ or $k = n-1$. (One could use calculus, for example, to see that there is only one critical point inside the interval, a local maximum at $k = n/2$.) In either case, $k(n-k) = n-1$ and the number of edges in the graph is

  $$\binom{n}{2} - (n-1) = \binom{n-1}{2}.$$

  The only example of a disconnected graph with $\binom{n-1}{2}$ edges is a clique of order $n-1$ together with an isolated vertex. Our argument shows that a graph with more edges than this must be connected, which is the desired result.

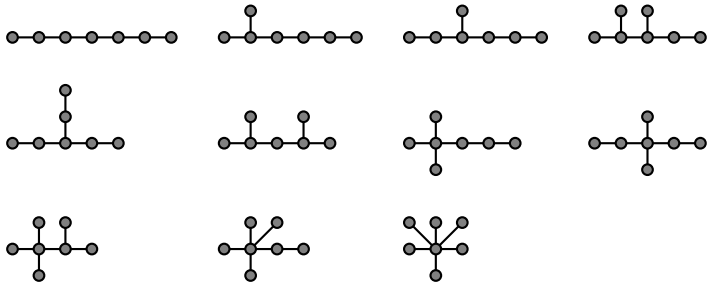
- #39
  Call a graph cubic if each vertex has degree equal to 3. The complete graph K_4 is the smallest example of a cubic graph. Find an example of a connected, cubic graph that does not have a Hamilton path.



- #44
  Which complete bipartite graphs $K_{m,n}$ have Hamilton cycles? Which have Hamilton paths?
  Suppose first that $v_1 - v_2 - \cdots - v_{m+n} - v_1$ is a Hamiltonian cycle. Then first of all, $m + n$ is even, since $K_{m,n}$ is bipartite and only contains cycles of even length. Moreover, every odd labelled vertex is in part A and every even labelled vertex in part B. Since every vertex is in the cycle, this gives that $|A| = |B|$, or $m = n$. Finally, if there is a cycle at all then m and n must be at least 2. So $K_{m,n}$ has a Hamiltonian cycle if and only if $m = n$ and $m, n \geq 2$.

  By essentially the same reasoning, $K_{m,n}$ has a Hamiltonian path if and only if $|m - n| \leq 1$ and $m, n \geq 1$.

- #56
  Grow all the nonisomorphic trees of order 7.



- #62
  Prove that, if a tree has a vertex of degree p, then it has at least p pendent vertices.

Suppose that the tree T has n vertices, with degree sequence $d_1, d_2, \ldots, d_n$. Since T has $n-1$ edges, we have

$$d_1 + d_2 + \cdots + d_n = 2(n-1) = 2n - 2.$$

Every vertex has degree at least one. Suppose without loss of generality that $d_1 = p$. Then

$$d_2 + d_3 + \cdots + d_n = 2n - 2 - p.$$

Suppose that m of the remaining vertices are pendent, i.e. have degree 1, and the remaining $n-1-m$ vertices in this sum all have degree ≥ 2. Then the total of these vertex degrees is at least $m + 2(n-1-m)$, so we have

$$m + 2(n-1-m) \leq 2n - 2 - p.$$

After some cancellation, it follows that $m \geq p$, as desired.