Name: _____

Math 4575 : midterm #2
Spring 2018

No calculators or notes are allowed.

Please keep your phones off or in airplane mode until after the exam.

Before you start, please check to make sure that you have six problems.

You have 55 minutes to complete the exam, 1:50–2:45pm.

Relax. Have fun.

1. Solve the recurrence $h_n = 3h_{n-1} - 2h_{n-2}, (n \geq 2)$ with initial values $h_0 = 2$, $h_1 = 5$.

   The characteristic polynomial is $x^2 - 3x + 2 = 0$, with roots $x = 1, 2$. So the general solution is given by

   $$h_n = c_1 2^n + c_2 1^n = c_1 2^n + c_2,$$

   for some constants c_1, c_2.

   Putting in the initial conditions gives

   $$\begin{aligned} 2 &= c_1 + c_2 \\ 5 &= 2c_1 + c_2 \end{aligned}$$

   So $c_1 = 3$, $c_2 = -1$, and

   $$h_n = 3 \times 2^n - 1.$$

2. (a) Six people are on a bus, and the bus will make three more stops. Everyone must get off at one of the three stops. In how many ways can this happen?

This is just counting functions $[6] \to [3]$, so there are

$$3^6$$

ways this can happen.

(b) In how many ways can this happen, if at least one person must get off at each stop?

(solution #1) This is counting surjective functions $[6] \to [3]$. First, partition $[6]$ into 3 nonempty subsets, $\left\{ {6 \atop 3} \right\}$ ways, and then map each nonempty subset to a different element 1, 2, or 3, giving a factor of $3!$ ways. In total, there are

$$\left\{ {6 \atop 3} \right\} 3!$$

ways to do this.

(solution #2) One can also use inclusion-exclusion counting. Let A denote the set of all functions $f : [6] \to [3]$. For $i = 1, 2, 3$, let S_i denote the set of functions f such that for every x, $f(x) \neq i$. This gives an equivalent answer

$$3^6 - \binom{3}{1} 2^6 + \binom{3}{2} 1^6.$$

3. Determine a formula for the number of ways h_n to color the squares of a $1 \times n$ chessboard with white, red, and green, if the number of red squares must be even and there must be at least one green square.

We use exponential generating functions. Let

$$f(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}.$$

Then $f(x) = f_1(x)f_2(x)f_3(x)$, where

$$f_1(x) = 1 + x + \frac{x^2}{2!} + \cdots = e^x,$$

$$f_2(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = (e^x + e^{-x})/2,$$

and

$$f_3(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x - 1.$$

Then

$$\begin{aligned} f_1(x)f_2(x)f_3(x) &= e^x \left(e^x + e^{-x}\right)\left(e^x - 1\right)/2 \\ &= e^x \left(e^{2x} - e^x + 1 - e^{-x}\right)/2 \\ &= \frac{1}{2}\left(e^{3x} - e^{2x} + e^x - 1\right). \end{aligned}$$

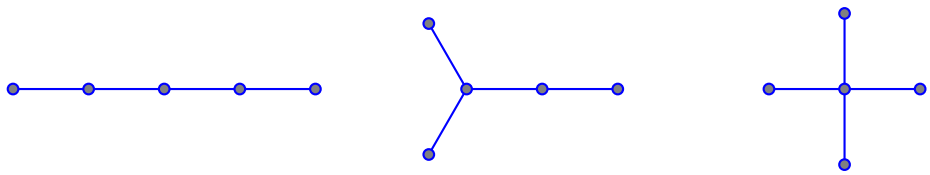
So

$$h_n = \left(3^n - 2^n + 1\right)/2$$

for $n \geq 1$, and $h_0 = 0$.

4. (a) How many isomorphism types are there of trees on 5 vertices? Draw a picture of each type.

There are three isomorphism types, as follows.



(b) How many isomorphism types are there of graphs on 12 vertices with degree sequence $(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$? Describe all of them.

Such a graph must be collection of disjoint cycles. Knowing how many cycles and of each length is enough to tell the isomorphism type. This is the same as the number of partitions of 12 into parts of size at least 3. We have

$$\begin{aligned} 12 &= 12 \\ &= 9 + 3 \\ &= 8 + 4 \\ &= 7 + 5 \\ &= 6 + 6 \\ &= 6 + 3 + 3 \\ &= 5 + 4 + 3 \\ &= 4 + 4 + 4 \\ &= 3 + 3 + 3 + 3 \end{aligned}$$

So there are 9 isomorphism types.

5. (a) Show that in a finite graph, the number of vertices of odd degree must be even.

Count the number of incident vertex-edge pairs, i.e. the number of $(v, e) \in V \times E$ with v an endpoint of e, in two different ways.
Counting by vertices, we get $\sum_{i=1}^{n} \deg(v_i)$. Counting be edges, we get $2e$. So

$$\sum_{i=1}^{n} \deg(v_i) = 2e.$$

Considering both sides mod 2 gives the result.

(b) Show that a finite tree with a vertex of degree $d \geq 2$ must have at least d vertices of degree 1.

Suppose that the tree T has n vertices, with degree sequence $d_1, d_2, \ldots, d_n$. Since T has $n-1$ edges, we have

$$d_1 + d_2 + \cdots + d_n = 2(n-1) = 2n - 2.$$

Every vertex has degree at least one. Suppose without loss of generality that $d_1 = p$. Then

$$d_2 + d_3 + \cdots + d_n = 2n - 2 - p.$$

Suppose that m of the remaining vertices have degree 1, and the remaining $n - 1 - m$ vertices in this sum all have degree ≥ 2. Then the total of these vertex degrees is at least $m + 2(n - 1 - m)$, so we have

$$m + 2(n - 1 - m) \leq 2n - 2 - p.$$

After some cancellation, it follows that $m \geq p$, as desired.

6. Let G be a graph with degree sequence $(d_1, d_2, \ldots, d_n)$. Prove that for every k with $1 \leq k \leq n-1$, we have

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

We interpret the left side as counting incident pairs (v, e), here v is a vertex in the first k vertices of G, and e is an edge containing v.

Now we try to count incident pairs (v, e), but by summing over edges rather than over vertices. Every edge counted in an incident pair is in one of two categories: either both its ends are in the first k vertices, or only one end. There are at most $\binom{k}{2}$ edges with both ends on first k vertices, and each such edge would get counted twice, once for each end of the edge, so these edges contribute at most $k(k-1)$ to the sum.

Every other edge has one end on one of the last $n-k$ vertices, and the other end on one of the first k vertices. For $i = k+1, \ldots, n$, the contribution of vertex i to the count of incident pairs can be at most k, since each edge must have its other end in the first k vertices. On the other hand we also note that its contribution can be at most d_i, since that is its total degree. Since both these bounds hold, we have that the contribution from vertex i is at most $\min\{k, d_i\}$.

Putting it all together, we have the desired inequality.