Math 4575 : HW #12

Due Wednesday, April 25 by 4pm

1. Let T be a tree on n vertices. Let $P_T(k)$ denote the chromatic polynomial of T. Show that
$$P_T(k) = k(k-1)^{n-1}.$$

   Consider induction on the number of vertices n. The claim holds for $n = 1$: there are k k-colorings of a single vertex.

   Now suppose it holds for trees on $n \geq 1$ vertices. A tree on $n+1 \geq 2$ vertices has at least one edge, so we know it has a pendent vertex v, i.e. a vertex of degree one. If you delete v, the remaining tree has $k(k-1)^{n-1}$ colorings, by the induction hypothesis. The vertex v is only connected to one other vertex in the tree, so the coloring of the smaller tree can be extended to include v in $k-1$ different ways—the only thing that is forbidden is giving v the same color as its neighbor. Since this holds for every coloring, the total number of colorings for a tree on $n+1$ vertices is
$$k(k-1)^{n-1}(k-1) = k(k-1)^n,$$
   as desired.

2. Let $G = C_n$ be a cycle of length n. Let $P_G(k)$ denote the chromatic polynomial of G. Show that
$$P_G(k) = (k-1)^n + (-1)^n(k-1).$$

   We proceed by induction on n, where the base case is the smallest cycle $n = 3$. The cycle $G = C_3$ is the same as the complete graph $G = K_3$, so since every vertex is connected to every other vertex, the chromatic polynomial is easily seen to be
$$P_G(k) = k(k-1)(k-2).$$
   A little rearranging the terms shows that this is equal to
$$(k-1)^3 + (-1)^3(k-1),$$
   so this completes the base case.

   Now suppose that the propositions holds for a given $n \geq 3$, and let $G = C_{n+1}$. We have the deletion-contraction relation:
$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$
   $G - e$ is a path with $n+1$ vertices, which is a special case of a tree on n vertices. So by the pervious problem, we have that
$$P_{G-e}(k) = k(k-1)^n.$$

On the other hand, G/e is a cycle with n vertices, so by the induction hypothesis we have that

$$P_{G/e}(k) = (k-1)^n + (-1)^n(k-1).$$

Combining, we have that

$$\begin{aligned} P_G(k) &= k(k-1)^n - ((k-1)^n + (-1)^n(k-1)) \\ &= k(k-1)^n - (k-1)^n + (-1)^{n+1}(k-1) \\ &= (k-1)^{n+1} + (-1)^{n+1}(k-1), \end{aligned}$$

so the formula holds for all $n \geq 3$ by induction.

3. Show that $P(k) = k^4 - 4k^3 + 3k^2$ is not the chromatic polynomial of any graph.

   Indeed! $P(2) < 0$, but the number of 2-colorings of a graph can not be negative.

4. Show that if a graph G has e edges then

$$\chi(G) \leq \sqrt{2e + \frac{1}{4}} + \frac{1}{2}.$$

   Color G with the minimum number of colors. The coloring is a partition of the vertices of G into independent sets $V_1 \amalg V_2 \amalg \cdots \amalg V_k$ where $k = \chi(G)$. Between every pair of vertex subsets V_i, V_j with $i \neq j$, there must be at least one edge. Otherwise we could merge vertex subsets V_i and V_j into a single color class, and then we have colored with one fewer color, a contradiction. Moreover, all these edges are distinct, as i, j ranges over all $\binom{k}{2}$ pairs.

   So have that

$$e \geq \binom{k}{2}.$$

   Completing the square and rearranging of terms gives that

$$k \leq \sqrt{2e + \frac{1}{4}} + \frac{1}{2}.$$

For the next two problems, define the Cartesian product $G \square H$ of two graphs G and H by:

$$V(G \square H) = \{(g, h) \mid g \in G, h \in H\}.$$

Use the notation $x \sim y$ to denote that vertex x is adjacent to vertex y. With this notation, the edges in $G \square H$ are defined as follows. We have $(g, h) \sim (g', h')$ whenever either

(a) $g = g'$ and $h \sim h'$ or

(b) $g \sim g'$ and $h = h'$.

For example, $K_2 \square K_2$ is isomorphic to C_4.

5. Show that $K_2 \square K_2 \square \ldots \square K_2$, the product of d copies of K_2, is isomorphic to the hypercube graph Q_d, defined in HW #11.

   We proceed by induction on d. The case $d = 1$ holds, since $K_2 = Q_1$—they are both the graph with two vertices and one edge.

   Suppose it holds for some $d \geq 1$. Let $H = K_2 \square \ldots \square K_2$, where there are $d+1$ copies of K_2 in the product. Then, by the induction hypothesis, H is isomorphic to $Q_d \square K_2$. So we are done if we can show that $Q_d \square K_2$ is isomorphic to Q_{d+1}.

   We denote a vertex in $Q_d \square K_2$ by $((x_1, x_2, \ldots, x_d), v)$ where $(x_1, x_2, \ldots, x_d)$ is a binary sequence of length d and v is one of the two vertices of K_2, labelled either 0 or 1.

   The define a function $f : V(Q_d \square K_2) \to V(Q_{d+1})$ by

$$f\left((x_1, x_2, \ldots, x_d), v\right) = (x_1, x_2, \ldots, x_d, v).$$

   This is actually a function. The main thing to notice is that $(x_1, x_2, \ldots, x_d, v)$ is a binary sequence of length $d+1$, since v is labelled either 0 or 1.

   The fact that f is an isomorphism of graphs follows immediately from the definitions of hypercube graph and Cartesian product.

6. Show that for every pair of graphs G, H, we have

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\}.$$

   First we show that

$$\chi(G \square H) \leq \max\{\chi(G), \chi(H)\}.$$

   Let $k = \max\{\chi(G), \chi(H)\}$. Let $f_1 : V(G) \to [k]$ and $f_2 : V(H) \to [k]$ be proper k-colorings. These both exist because we chose k to be the maximum chromatic number. We can k-color both graphs, even if all k colors don't necessarily get used for one graph.

   Now define a coloring $f : G \square H \to [k]$ by setting

$$f(g, h) \equiv f_1(g) + f_2(h) \pmod{k}.$$

   (Note that we choose for the representatives for the k congruence classes $1, 2, \ldots, k$, rather than $0, 1, \ldots, k = 1$, if we are considering our colors $1, 2, \ldots, k$.)

   We check that f is a proper coloring, which is just to say that if $(g, h) \sim (g', h')$, then $f(g, h) \neq f(g', h')$. If $(g, h) \sim (g', h')$ then the definition of Cartesian product of two graphs tells us that there are two possibilities. Suppose that $g = g'$ and $h \sim h'$. Then $f_1(g) = f_1(g')$, but $f_2(h) \neq f_2(h')$. Then we can conclude that $f_1(g) + f_2(h) \neq f_1(g') + f_2(h')$. The other case is similar.

Now, it is not hard to see that

$$\chi(G\Box H) \geq \max\{\chi(G), \chi(H)\}$$

since $G\Box H$ contains isomorphic copies of G and H.