Annotated bibliography of Matthew Kahle

Stochastic topology

1. M. Kahle. The neighborhood complex of a random graph. *J. Combin. Theory Ser. A* 114 (2007), no. 2, 380–387.

   For a graph G, the neighborhood complex $N[G]$ is the simplicial complex on the vertices of G having all subsets of vertices sharing a common neighbor as its faces. It is a well known result of Lovász that if $N[G]$ is k-connected, then the chromatic number of G is at least $k+3$. We prove that the connectivity of the neighborhood complex of a random graph is tightly concentrated, with high probability between $1/2$ and $2/3$ of the expected clique number. We also show that the number of dimensions of nontrivial homology is with high probability small, $O(\log d)$, compared to the expected dimension d of the complex itself. One motivation for this work was to understand how well topological lower bounds on chromatic number perform for "typical" graphs.

2. M. Kahle. Topology of random clique complexes. *Discrete Math.*, 309 (2009), no. 6, 1658–1671.

   Erdős and Rényi showed that if $p \gg \log n/n$ then the random graph $G(n,p)$ is connected with high probability, and if $p \ll \log n/n$ then $G(n,p)$ is disconnected with high probability, as $n \to \infty$. The clique complex $X(H)$ of a graph H is the simplicial complex with all complete subgraphs of H as its faces. We study the clique complex of $G(n,p)$, denoted $X(n,p)$. For $k \geq 1$, we show that if $p \ll n^{-1/k}$ or $p \gg n^{1/(2k+1)}$, then the kth homology $H_k(X(n,p)) = 0$ with high probability, and if $n^{-1/(k+1)} \ll p \ll n^{-1/k}$, then with high probability $H_k(X(n,p)) \neq 0$. We also give estimates for the expected rank of homology and exhibit explicit nontrivial classes in the regime where homology is nontrivial. These estimates suggest the "bouquet-of-spheres conjecture", that random clique complexes are homotopy equivalent to wedge sums of spheres with high probability.

3. E. Babson, C. Hoffman, and M. Kahle. The fundamental group of random 2-complexes. *J. Amer. Math. Soc.* 24 (2011), no. 1, 1–28.

   We study Linial–Meshulam random 2-complexes, which are two-dimensional analogues of Erdős-Rényi random graphs. We find the threshold for simple connectivity to be roughly $p = n^{-1/2}$. This is much larger than the vanish threshold for vanishing for homological connectivity, which was shown by Linial and Meshulam to be $p = 2\log n/n$. We use a variant of Gromov's local-to-global theorem for linear isoperimetric inequalities to show that when $p = O(n^{-1/2-\varepsilon})$,

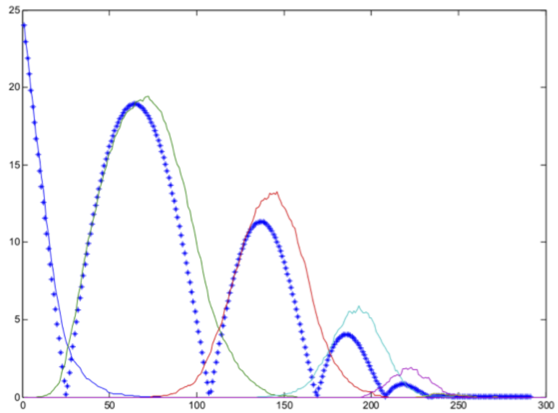

Figure 1: The Betti numbers of a random clique complex on $n = 25$ vertices are shown with the continuous curves. The blue stars represent the absolute value of the expected Euler characteristic. The fact that the curves align so well is closely related to the fact that homology tends to concentrate in one degree.

with high probability the fundamental group is word hyperbolic. Along the way we classify the homotopy types of sparse 2-dimensional simplicial complexes and establish isoperimetric inequalities for such complexes. These intermediate results do not involve randomness and may be of independent interest.

4. M. Kahle. Random geometric complexes. *Discrete Comput. Geom.*, 45 (2011), no. 3, 553–573.

   We study the expected topological properties of Čech and Vietoris–Rips complexes built on i.i.d. random points in $\mathbb{R}^d$. We find higher dimensional analogues of known results for connectivity and component counts for random geometric graphs. However, homology H_k is not monotone when $k \geq 1$. In particular, for every $k \geq 1$ we show the existence of two thresholds, one where homology passes from vanishing to nonvanishing, and another where it passes back to vanishing. We give asymptotic formulas for the expectation of the Betti numbers in the sparser regimes, and bounds in the denser regimes. The main technical contribution of the article is in the application of discrete Morse theory in geometric probability.

5. D. Dotterrer and M. Kahle. Coboundary expanders. *J. Topol. Anal.* 4 (2012), no. 4, 499–514.

   We describe a natural topological generalization of edge expansion for graphs to regular CW complexes and prove that this property holds with high probability for certain random complexes.

6. M. Kahle and E. Meckes. Limit theorems for Betti numbers of random simplicial complexes. *Homology, Homotopy Appl.* 15(2) (2013), 343–374.

   In this article we establish Poisson and normal approximation theorems for Betti numbers of a different models of random simplicial complex: Erdős-Rényi random clique complexes, random Vietoris–Rips complexes, and random Čech complexes. These results may be of practical interest in topological data analysis.

   There was a mistake in in our main proof. The proof is corrected in the erratum.

   M. Kahle and E. Meckes. Erratum to "Limit theorems for Betti numbers of random simplicial complexes". *Homology Homotopy Appl.* 18 (2016), no. 1, 129–142.

7. M. Kahle. Sharp vanishing thresholds for cohomology of random flag complexes. *Ann. of Math.* 179 (2014), 1085–1107.

   For every $k \geq 1$, the kth cohomology group $H^k(X,\mathbb{Q})$ of the random flag complex $X \sim X(n,p)$ passes through two phase transitions: one where it appears, and one where it vanishes. We describe the vanishing threshold and show that the threshold is sharp. This is a high-dimensional analogue of the Erdős–Rényi theorem characterizing the threshold for connectivity of the random graph.

   In particular, we show that if

$$p = \left(\frac{\left(\frac{k}{2}+1\right)\log n + \left(\frac{k}{2}\right)\log\log n + c}{n} \right)^{1/(k+1)},$$

   where $k \geq 1$ and $c \in \mathbb{R}$ are fixed, then

$$\mathbb{P}\left[H^k(X,\mathbb{Q}) = 0\right] \to \exp\left(-\frac{\left(\frac{k}{2}+1\right)^{k/2}}{(k+1)!} e^{-c} \right)$$

   The case $k = 0$ corresponds to the Erdős–Rényi random graph.

   Combining with earlier results, we obtain as a corollary that for every $k \geq 3$, there is a regime in which the random flag complex is rationally homotopy equivalent to a bouquet of k-dimensional spheres.

8. M. Davis and M. Kahle. Random graph products of finite groups are rational duality groups. *J. Topol.*, 7 (2014), 589–606.

   Given an edge-independent random graph $G(n,p)$, we determine various facts about the cohomology of graph products of groups for the graph $G(n,p)$. In particular, the random graph product of a sequence of finite groups is a rational duality group with probability tending to 1 as n goes to infinity. This includes random right-angled Coxeter groups as a special case.

9. M. Kahle and B. Pittel. Inside the critical window for cohomology of random k-complexes. *Random Structures & Algorithms* 48 (2016), no. 1, 102–124.

   We prove sharper versions of theorems of Linial–Meshulam and Meshulam–Wallach which describe the behavior for $(\mathbb{Z}/2)$-cohomology of a random k-dimensional simplicial complex within a narrow transition window. In particular, we show that within this window the $(k-1)$st Betti number is in the limit Poisson distributed. For $k = 2$ we also prove that in an accompanying growth process, with high probability, first cohomology vanishes exactly at the moment when the last isolated $(k-1)$-simplex gets covered by a k-simplex.

10. C. Hoffman, M. Kahle, and E. Paquette. The threshold for integer homology in random d-complexes. *Discrete Comput. Geom.* 57 (2017), no. 4, 810–823.

   Let $Y \sim Y_d(n,p)$ denote the Bernoulli random d-dimensional simplicial complex. We answer a question of Linial and Meshulam from 2003, showing that the threshold for vanishing of homology $H_{d-1}(Y,\mathbb{Z})$ is less than $40d(d+1)\log n/n$. This bound is tight, up to a constant factor which depends on d.

11. O. Bobrowski, M. Kahle, and P. Skraba. Maximally persistent cycles in random geometric complexes. *Ann. Appl. Probab.* 27 (2017), no. 4, 2032–2060.

   We study maximally persistent cycles of degree k in persistent homology, for a either the Čech or the Vietoris–Rips filtration built on a uniform Poisson process of intensity n in the unit cube $[0,1]^d$. This is a natural way of measuring the largest "k-dimensional hole" in a random point set in $\mathbb{R}^d$, where $d = 2$ is fixed and $n \to \infty$. This problem is in the intersection of geometric probability and algebraic topology, and is naturally motivated by a probabilistic view of topological inference. We show that for fixed $d \geq 2$ and $1 \leq k \leq d-1$, the maximally persistent cycle has (multiplicative) persistence of order

   $$\Theta\left(\left(\frac{\log n}{\log\log n}\right)^{1/k}\right)$$

   with high probability, characterizing its rate of growth as $n \to \infty$. The implied constants depend on k, d, and on whether we consider the Vietoris–Rips or Čech filtration.

12. D. Dotterrer, L. Guth, M. Kahle. 2-Complexes with Large 2-Girth. *Discrete Comput. Geom.* 59 (2018), no. 2, 383–412.

   The *2-girth* of a 2-dimensional simplicial complex X is the minimum size of a non-zero 2-cycle in $H_2(X,\mathbb{Z}/2)$. We consider the maximum possible girth of a complex with n vertices and m 2-faces. If $m = n^{2+\alpha}$ for $\alpha < 1/2$, then we show that the 2-girth is at most $4n^{2-2\alpha}$ and we prove the existence of complexes with 2-girth at least $c_{\alpha,\varepsilon}n^{2-2\alpha-\varepsilon}$. On the other hand, if $\alpha > 1/2$, the 2-girth is at most $C\alpha$. So there is a phase transition as α passes $1/2$. Our results depend on a new upper bound for the number of combinatorial types of triangulated surfaces with v vertices and f faces.

13. M. Kahle, F. Lutz, A. Newman, K. Parsons. Cohen–Lenstra heuristics for homology of random complexes. *Experimental Mathematics*, 29:3, 347–359, (2020).

    We study torsion in homology of the random d-complex $Y \sim Y_d(n,p)$ experimentally. Our experiments suggest that there is almost always a moment in the process where there is an enormous burst of torsion in homology $H_{d-1}(Y)$. This moment seems to coincide with the phase transition studied by Linial and Peled, where cycles in $H_d(Y)$ first appear with high probability.

    Our main study is the limiting distribution on the q-part of the torsion subgroup of $H_{d-1}(Y)$ for small primes q. We find strong evidence for a limiting Cohen–Lenstra distribution, where the probability that the q-part is isomorphic to a given q-group H is inversely proportional to the order of the automorphism group $|\mathrm{Aut}(H)|$. We also study the torsion in homology of the uniform random $\mathbb{Q}$-acyclic 2-complex. This model is analogous to a uniform spanning tree on a complete graph, but more complicated topologically since Kalai showed that the expected order of the torsion group is exponentially large in n^2. We give experimental evidence that in this model also, the torsion is Cohen–Lenstra distributed in the limit.

14. M. Kahle and A. Newman. Topology and geometry of random 2-dimensional hypertrees. *Discrete Comput. Geom.* 67 (2022), no. 4, 1229–1244.

    A hypertree, or $\mathbb{Q}$-acyclic complex, is a higher-dimensional analogue of a tree. We study random 2-dimensional hypertrees according to the determinantal measure suggested by Lyons, where each hypertree T is weighted by $|H_1(T)|^2$. We are especially interested in their topological and geometric properties. We show that with high probability, a random 2-dimensional hypertree T is apsherical, i.e. that it has a contractible universal cover. We also show that with high probability the fundamental group $\pi_1(T)$ is hyperbolic and has cohomological dimension 2.

15. M. Kahle, E. Paquette, and É. Roldán. Topology of random 2-dimensional cubical complexes. *Forum Math. Sigma* 9 (2021), Paper No. e76, 24 pp.

    We study a natural model of random 2-dimensional cubical complex which is a subcomplex of an n-dimensional cube, and where every possible square 2-face is included independently with probability p. Our main result exhibits a sharp threshold $p = 1/2$ for homology vanishing as $n \to \infty$. This is a 2-dimensional analogue of the Burtin and Erdős–Spencer theorems characterizing the connectivity threshold for random graphs on the 1-skeleton of the n-dimensional cube.

    Our main result can also be seen as a cubical counterpart to the Linial–Meshulam theorem for random 2-dimensional simplicial complexes. However, the models exhibit strikingly different behaviors. We show that if $p > 1 - \sqrt{1/2} \approx 0.2929$, then with high probability the fundamental group is a free group with one generator for every maximal 1-dimensional face. As a corollary, homology vanishing and simple connectivity have the same threshold, even in the strong "hitting time" sense. This is in contrast with the simplicial case, where the thresholds are far apart.

The main proof depends on an iterative algorithm for contracting cycles — we show that with high probability the algorithm rapidly and dramatically simplifies the fundamental group, converging after only a few steps.

16. P. Duncan, M. Kahle, and B. Schweinhart. Homological percolation on a torus: plaquettes and permutohedra. *submitted*. (arXiv:2011.11903).

    We study higher-dimensional homological analogues of bond percolation on a square lattice and site percolation on a triangular lattice.

    By taking a quotient of certain infinite cell complexes by growing sublattices, we obtain finite cell complexes with a high degree of symmetry and with the topology of the torus $\mathbb{T}^d$. When random subcomplexes induce nontrivial i-dimensional cycles in the homology of the ambient torus, we call such cycles *giant*. We show that for every i and d there is a sharp transition from nonexistence of giant cycles to giant cycles spanning the homology of the torus.

    We also prove convergence of the threshold function to a constant in certain cases. In particular, we prove that $p_c = 1/2$ in the case of middle dimension $i = d/2$ for both models. This gives finite-volume high-dimensional analogues of Kesten's theorems that $p_c = 1/2$ for bond percolation on a square lattice and site percolation on a triangular lattice.

17. A. Ababneh and M. Kahle. Maximal persistence in random clique complexes.

    We study the persistent homology of an Erdős–Rényi random clique complex filtration on n vertices. Here, each edge e appears at a time $p_e \in [0,1]$ chosen uniform randomly in the interval, and the *persistence* of a cycle σ is defined as p_2/p_1, where p_1 and p_2 are the birth and death times of the cycle respectively. We show that for fixed $k \geq 1$, with high probability the maximal persistence of a k-cycle is of order roughly $n^{1/k(k+1)}$. These results are in sharp contrast with the random geometric setting where earlier work by Bobrowski, Kahle, and Skraba shows that for random Čech and Vietoris–Rips filtrations, the maximal persistence of a k-cycle is much smaller, of order $(\log n/\log\log n)^{1/k}$.

Topological statistical mechanics

1. M. Kahle. Sparse locally-jammed disk packings. *Ann. Comb.* 16(4) (2012), 773–780.

   We construct arbitrarily sparse locally-jammed packings of non-overlapping congruent disks in various finite area regions—in particular, we give constructions for the square, hexagon, and for certain flat tori.

2. G. Carlsson, J. Gorham, M. Kahle, and J. Mason. Computational topology for configuration spaces of hard disks. *Phys. Rev. E*, 85 (2012).

   We explore the topology of configuration spaces of hard disks experimentally, and show that several changes in the topology can already be observed with a small number of particles. The results illustrate a theorem of Baryshnikov, Bubenik, and Kahle that critical points correspond to configurations of disks with balanced mechanical stresses, and suggest conjectures about the asymptotic topology as the number of disks tends to infinity.

3. Y. Baryshnikov, P. Bubenik, and M. Kahle. Min-type Morse theory for configuration spaces of hard spheres. *Int. Math. Res. Notices* 9 (2014), 2577–2592.

   We study configuration spaces of hard spheres in a bounded region. We develop a general Morse-theoretic framework, and show that mechanically balanced configurations play the role of critical points. As an application, we find the precise threshold radius for a configuration space to be homotopy equivalent to the configuration space of points.

4. H. Alpert, M. Kahle, and R. MacPherson (with appendix by Ulrich Bauer and Kyle Parsons). Configuration spaces of disks in an infinite strip. *Journal of Applied & Computational Topology* 5 (2021), 357–390.

   We study the topology of the configuration spaces $C(n,w)$ of n hard disks of unit diameter in an infinite strip of width w. We describe ranges of parameter or "regimes", where homology $H_j[C(n,w)]$ behaves in qualitatively different ways.

   We show that if $w \geq j+2$, then the homology $H_j[C(n,w)]$ is isomorphic to the homology of the configuration space of points in the plane, $H_j[C(n,\mathbb{R}^2)]$. The Betti numbers of $C(n,\mathbb{R}^2)$ were computed by Arnol'd, and so as a corollary of the isomorphism, $\beta_j[C(n,w)]$ is a polynomial in n of degree $2j$.

   On the other hand, we show that if $2 \leq w \leq j+1$, then $\beta_j[C(n,w)]$ grows exponentially with n. Most of our work is in carefully estimating $\beta_j[C(n,w)]$ in this regime.

   We also illustrate, for every n, the homological "phase portrait" in the (w,j)-plane— the parameter values where homology $H_j[C(n,w)]$ is trivial, nontrivial, and isomorphic with $H_j[C(n,\mathbb{R}^2)]$. Motivated by the notion of phase transitions for hard-spheres systems, we discuss these as the "homological solid, liquid, and gas" regimes. See Figure 2.

5. H. Alpert, U. Bauer, M. Kahle, R. MacPherson, and K. Spendlove. Homology of configuration spaces of hard squares in a rectangle. *to appear in Algebraic & Geometric Topology*. (arXiv:2010.14480).

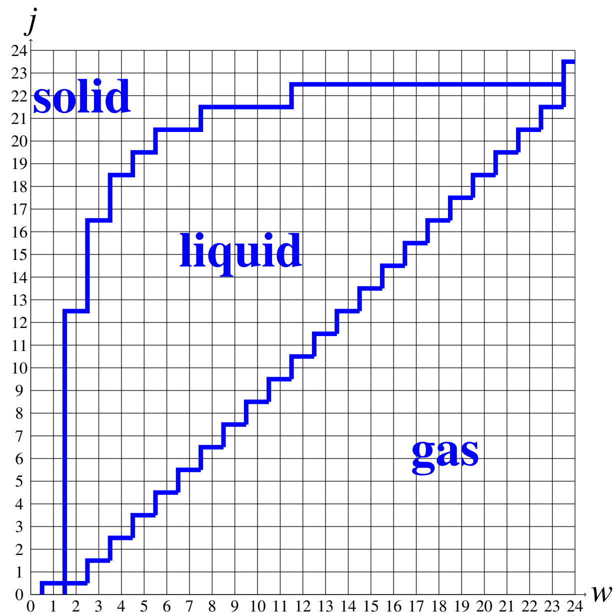

Figure 2: We describe the phase portrait of the homological solid, liquid, and gas regimes for every n. We illustrate here the case $n = 24$.

We study ordered configuration spaces $C(n;p,q)$ of n hard squares in a $p \times q$ rectangle, a generalization of the well-known "15 Puzzle". Our main interest is in the topology of these spaces. Our first result is to describe a cubical cell complex and prove that is homotopy equivalent to the configuration space. We then focus on determining for which n, j, p, and q the homology group $H_j[C(n;p,q)]$ is nontrivial. We prove three homology-vanishing theorems, based on discrete Morse theory on the cell complex. Then we describe several explicit families of nontrivial cycles, and a method for interpolating between parameters to fill in most of the picture for "large-scale" nontrivial homology.

6. H. Alpert, M. Kahle, R. MacPherson. Asymptotic Betti numbers for hard squares in the homological liquid regime. *submitted*. (arXiv:2207.13139).

   We study configuration spaces $C(n;p,q)$ of n ordered unit squares in a p by q rectangle. Our goal is to estimate the Betti numbers for large n, j, p, and q. We consider sequences of area-normalized coordinates, where $\left(\frac{n}{pq}, \frac{j}{pq}\right)$ converges as n, j, p, and q approach infinity. For every sequence that converges to a point in the "feasible region" in the (x,y)-plane, we show that the factorial growth rate of the Betti numbers is the same as the factorial growth rate of $n!$. This implies that (1) the Betti numbers are vastly larger than for the configuration space of n ordered points in the plane, which have the factorial growth rate of $j!$, and (2)

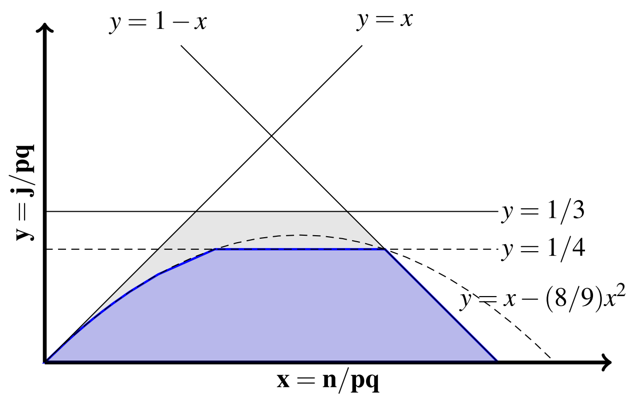

Figure 3: We proved that if $H_j[C(n;p,q)] \neq 0$ and $(x,y) = (n/pq, j/pq)$, then (x,y) is in the shaded region $0 \leq y \leq \min\{x, 1-x, 1/3\}$. The blue region is what we call the feasible region $\mathscr{R}$. In another paper, we show that for every point in the interior of the region, Betti numbers grow factorially fast.

every point in the feasible region is eventually in the homological liquid regime.

Random graphs

1. C. Hoffman, M. Kahle, and E. Paquette. Spectral gaps of random graphs and applications. *Int. Math. Res. Notices* (2021), 8353–8404.

   We study the spectral gap of the Erdős–Rényi random graph through the connectivity threshold. In particular, we show that for any fixed $\delta > 0$, if $p \geq (1/2+\delta)\log n/n$, then the normalized graph Laplacian of $G(n,p)$ has its nonzero eigenvalues tightly concentrated around 1. This is a strong expander property. We estimate both the decay rate of the spectral gap to 1 and the failure probability, up to a constant factor. We also show that the 1/2 in the above is optimal, and that if $pc\log n$ and $c < 1/2$, then there are eigenvalues of the Laplacian restricted to the giant component that are separated from 1. We then describe several applications of our spectral gap results to stochastic topology and geometric group theory. These all depend on Garland's method, a kind of spectral geometry for simplicial complexes. The following can all be considered to be higher-dimensional expander properties.

   – First, we exhibit a sharp threshold for the fundamental group of the Bernoulli random 2-complex to have Kazhdan's property (T). We also obtain slightly more information and can describe the large-scale structure of the group just before the (T) threshold. In this regime, the random fundamental group is with high probability the free product of a (T) group with a free group, where the free group has one generator for every isolated edge. The (T) group plays a role analogous to that of a "giant component" in percolation theory.

- Next we give a new, short, self-contained proof of the Linial–Meshulam–Wallach theorem, identifying the cohomology-vanishing threshold of random d-dimensional simplicial complexes. Since we use spectral techniques, it only holds for $\mathbb{Q}$ coefficients. However, it is sharp from a probabilistic point of view, providing hitting-time type results and limiting Poisson distributions inside the critical window. It is also a new method of proof, circumventing the combinatorial complications of cocycle counting.
- The spectral gap results from this article were also applied to obtain sharp cohomology-vanishing thresholds in every dimension for the random clique complex.

2. M. Kahle and F. Martinez-Figueroa. The chromatic number of random Borsuk graphs. *Random Structures & Algorithms* (2020), Vol 56. Issue 3. 838–850.

   We study a model of random graph where vertices are n i.i.d. uniform random points on the unit sphere S^d in $\mathbb{R}^{d+1}$, and a pair of vertices is connected if the Euclidean distance between them is at least $2-\varepsilon$. We are interested in the chromatic number of this graph as n tends to infinity.

   It is not too hard to see that if $\varepsilon > 0$ is small and fixed, then the chromatic number is $d+2$ with high probability. We show that this holds even if $\varepsilon \to 0$ slowly enough. We quantify the rate at which ε can tend to zero and still have the same chromatic number. The proof depends on combining topological methods (namely the Lyusternik–Schnirelman–Borsuk theorem) with geometric probability arguments. The rate we obtain is best possible, up to a constant factor — if $\varepsilon \to 0$ faster than this, we show that the graph is $(d+1)$-colorable with high probability.

3. M. Kahle, M. Tian, and Y. Wang. Local cliques in ER-perturbed random geometric graphs. *ISAAC (International Symposium on Algorithms and Computation)* (2019).

   Let $G^*_{\mathscr{X}}$ be a random geometric graph sampled from a nice measure on a metric space $\mathscr{X} = (X,d)$. The input observed graph $\widehat{G}(p,q)$ is generated by removing each existing edge from $G^*_{\mathscr{X}}$ with probability p, while inserting each non-existent edge to $G^*_{\mathscr{X}}$ with probability q. We refer to such random p-deletion and q-insertion as *ER-perturbation*.

   In this paper we consider a localized version of the classical notion of clique number for this type of random graphs. Specifically, we study the *edge clique number* for each edge in a graph, defined as the size of the largest clique(s) in the graph containing that edge. Given an ER-perturbed random geometric graph, we show that the edge clique number presents two fundamentally different types of behaviors, depending on which type of randomness it is generated from.

Our main interest is an application to a certain metric reconstruction problem.

4. S. Fadnavis, M. Kahle, and F. Martinez-Figueroa. Warmth and mobility of random graphs. *submitted*. (arXiv:1009.0792).

   A graph homomorphism from the rooted d-branching tree $\phi : T^d \to H$ is said to be *cold* if the values of ϕ for vertices arbitrarily far away from the root can restrict the value of ϕ at the root. *Warmth* is a graph parameter that measures the non-existence of cold maps. We study warmth of random graphs $G(n,p)$, and for every $d \geq 1$, we exhibit a nearly-sharp threshold for the existence of cold maps. As a corollary, for $p = O(n^{-\alpha})$ warmth of $G(n,p)$ is concentrated on at most two values. As another corollary, a conjecture of Lovász relating mobility to chromatic number holds for "almost all" graphs. Finally, our results suggest new conjectures relating graph parameters coming from statistical physics with graph parameters coming from equivariant topology.

5. M. Kahle, M. Tian, Y. Wang. On the clique number of noisy random geometric graphs. *submitted*, 2019.

   Let G_n be a random geometric graph, and then for $q,p \in [0,1)$ we construct a *(q,p)-perturbed noisy random geometric graph $G_n^{q,p}$* where each existing edge in G_n is removed with probability q, while and each non-existent edge in G_n is inserted with probability p. We give asymptotically tight bounds on the clique number $\omega\left(G_n^{q,p}\right)$ for several regimes of parameter.

6. B. Braun, K. Bruegge, M. Kahle. Facets of Random Symmetric Edge Polytopes, Degree Sequences, and Clustering. *submitted*. (arXiv:2204.07239).

   Symmetric edge polytopes are lattice polytopes associated with finite simple graphs that are of interest in both theory and applications. We investigate the facet structure of symmetric edge polytopes for various models of random graphs. For an Erdős–Rényi random graph, we identify a threshold probability at which with high probability the symmetric edge polytope shares many facet-supporting hyperplanes with that of a complete graph. We also investigate the relationship between the Watts-Strogatz clustering coefficient and the number of facets for a graph with either a fixed number of edges or a fixed degree sequence. We use well-known Markov Chain Monte Carlo sampling methods to generate empirical evidence that for a fixed degree sequence, higher Watts-Strogatz clustering in a connected graph corresponds to higher facet numbers in the associated symmetric edge polytope.

Discrete geometry

1. M. Kahle. A generalization of the chromatic number of the plane. *Geombinatorics* 10 (2000), no. 2, 69–74.

Define a graph denoted $(\mathbb{R}^d, 1)$ with vertices corresponding to points in Euclidean space $\mathbb{R}^d$, and with $\{x, y\}$ an edge whenever $d(x, y) = 1$. The Hadwiger–Nelson problem asks for the chromatic number of $(\mathbb{R}^2, 1)$. We define a class of graphs G_θ varying continuously with a parameter θ such that $G_0 = (\mathbb{R}^1, 1)$ and $G_{\pi/2} = (\mathbb{R}^2, 1)$. We study the chromatic numbers of these graphs, and identify ranges of θ where $\chi(G_\theta) = 2, 3, 4$.

2. M. Kahle. Scatters, unavoidable shapes, and crystallization. *Geombinatorics* 15 (2006), no. 3, 138–149.

   We study (n,k)*-scatters*, which are regular n-gon tiles arranged so that each tile shares edges with at least k others. To measure how much freedom there is in arranging scatters, we ask which shapes are unavoidable. It turns out that for a few choices of (n,k) there are infinite unavoidable shapes, but otherwise they are finite. We discuss the infinite case as an analogue of crystallization. The main result here is that besides the trivial situations when there's a unique scatter, there are only four instances of this. Scatters crystallize nontrivially just when $(n,k) = (5,3)$, $(7,3)$, $(10,4)$, or $(14,4)$.

3. M. Kahle. Points in a triangle forcing small triangles *Geombinatorics* 18 (2009), no. 3, 114–128.

   An old theorem of Alexander Soifer's is the following: Given five points in a triangle of unit area, there must exist some three of them which form a triangle of area $1/4$ or less. It is easy to see that this is not true if "five" is replaced by "four", but can the theorem be improved in any other way? We discuss in this article two different extensions of the original result. First, we allow the value of "small", $1/4$, to vary. In particular, our main result is to show that given five points in a triangle of unit area, then there must exist some three of them determining a triangle of area $6/25$ or less. Second, we put bounds on the minimum number of small triangles determined by n points in a triangle, and make a conjecture about the asymptotic right answer as n tends to infinity.

4. M. Kahle and B. Taha. New lower bounds on $\chi\left(\mathbb{R}^d\right)$ for $d = 8, \ldots, 12$. *Geombinatorics* 24 (2015), 109–116.

   We improve the best lower bounds on the chromatic number of Euclidean space $\chi\left(\mathbb{R}^d\right)$ in dimensions $d = 8, \ldots, 12$. The new results depend in part on extensive computer calculations.

5. M. Kahle and E. Roldán. Polyominoes with maximally many holes. *Geombinatorics* 29 (2019), no. 1, 5–20.

   What is the maximum number of holes that a polyomino with n tiles can enclose? Call this number $f(n)$. We show that if $n_k = \left(2^{2k+1} + 3 \cdot 2^{k+1} + 4\right)/3$ and $h_k = \left(2^{2k} - 1\right)/3$, then $f(n_k) = h_k$ for $k \geq 1$. We also give nearly matching upper and lower bounds for large n, showing in particular that $f(n) = (1 - o(1))\,n/2$.

6. M. Kahle, F. Martinez-Figueroa, and A. Soifer. A square-grid coloring problem. *Geombinatorics* 29 (2020), no. 4, 167–184.

   Suppose that $n \geq 2$, and we wish to plant k different types of trees in the squares of an $n \times n$ square grid. We can have as many of each type as we want. The only rule is that every pair of types must occur in an adjacent pair of squares somewhere in the grid. The question is: given n, what is the largest that k can be? Denote this number by $\Gamma(n)$, and call this the *complete coloring number* of the $n \times n$ grid. A little thought shows that $\Gamma(n) \leq 2n - 1$. The main question we are interested in is whether $\Gamma(n) = 2n - 1$ for every $n \geq 2$. Wer discuss why equality holds for all sufficiently large n, and we also show that $\Gamma(n) \geq 2n - 9$ for every $n \geq 2$,

Expository writings, technical reports, miscellanea

1. M. Kahle. Geometric random complexes. *Oberwolfach Report* No. 29/2008, p. 1626–1628.

   This is an Oberwolfach report on what eventually became the paper "Random geometric complexes".

2. M. Kahle. On Branko Grünbaum's 80th birthday. *Geombinatorics* 19 (2009), no. 2, 42–45.

   I recall some things I appreciated about Branko Grünbaum, as a teacher and mentor.

3. M. Kahle. The geometry of random spaces. Institute for Advanced Study (IAS), Princeton, NJ. *Institute Summer Letter*. Summer 2011.

   This is an article loosely related to some of my research in stochastic topology, written for a lay audience. It appeared in the Institute for Advanced Study's summer newsletter.

4. M. Kahle. Expansion properties of random simplicial complexes. *Oberwolfach Report* No. 24/2012, p. 1442-1445, DOI: 10.4171/OWR/2012/24.

   This is an Oberwolfach report on random simplicial complexes, and how they provide compelling "high-dimensional expanders" from both spectral and coboundary expansion points of view.

5. M. Kahle. Topology of random simplicial complexes: a survey. *AMS Contemp. Math.*, 620 (2014), 201–221.

   This expository article is based on a lecture from the Stanford Symposium on Algebraic Topology: Application and New Directions, held in honor of Gunnar Carlsson, Ralph Cohen, and Ib Madsen.

6. M. Kahle. Curiosities: Permutation Puzzles from Archimedes to the Rubik's Cube. Institute for Advanced Study (IAS), Princeton, NJ. *Institute Summer Letter*. Summer 2015.

   This is an article on mathematics of permutation puzzles, written for a lay audience. It appeared in the Institute for Advanced Study's summer newsletter.

7. M. Kahle. Configuration spaces of disks. *Oberwolfach Report* No. 45/2015, p. 2652, DOI: 10.4171/OWR/2015/45.

   This Oberwolfach report briefly overviews our first few papers in topological statistical mechanics.

8. M. Kahle. Book chapter on "Random Simplicial Complexes" in *Handbook of Discrete & Computational Geometry, 3rd Edition* (2017), CRC Press (25 pages).

   In this book chapter we overview topological and geometric properties of (abstract and geometric) random simplicial complexes. We introduce a few of the fundamental models in Section 1. We review high-dimensional expander-like properties of random complexes in Section 2. We discuss threshold behavior and phase transitions in Section 3, and Betti numbers and persistent homology in Section 4.

9. O. Bobrowski and M. Kahle. Topology of random geometric complexes: a survey. *Journal of Applied & Computational Topology*, 331–364 (2018).

   This is a survey of topology of random geometric simplicial complexes.

10. M. Kahle. Configuration spaces of disks in an infinite strip. *Oberwolfach Report* No. 39/2019, p. 2421–2424, DOI: 10.4171/OWR/2019/39.

    This is an Oberwolfach report about the paper with Alpert and MacPherson with the same title.

11. M. Kahle. Branko Grünbaum in many dimensions. *Geombinatorics* 28 (2019), no. 3, 140–146.

    This is a survey article, listing some open problems that I think Branko Grünbaum was interested in.