Sharp vanishing thresholds for cohomology of random flag simplicial complexes

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Algebraic Topology: Applications and New Directions

Random topology

"I predict a new subject of statistical topology. Rather than count the number of holes, Betti numbers, etc., one will be more interested in the distribution of such objects on noncompact manifolds as one goes out to infinity." — Isadore Singer, 2004.

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- Random simplicial complexes — Linial and Meshulam, 2006.

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- Why do so many groups / manifolds / simplicial complexes / etc. seem to have a certain topological property?

  E.g. many simplicial complexes and posets arising in combinatorics are known to be homotopy equivalent to wedges of spheres. But why does this happen so often?

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- Ramsey theory and extremal graph theory e.g. Erdős, ...
- Geometric group theory — e.g. Gromov, Żuk
- Expander graphs — e.g. Pinsker, Barzdin & Kolmogorov, etc.

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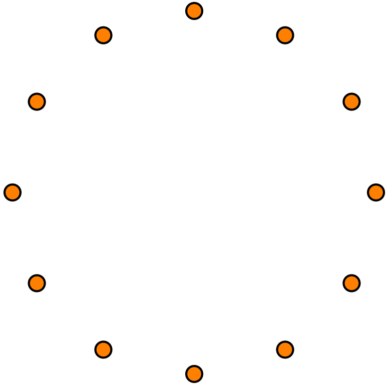
It is sometimes useful to think of $G(n, p)$ as a stochastic process.

Random graphs



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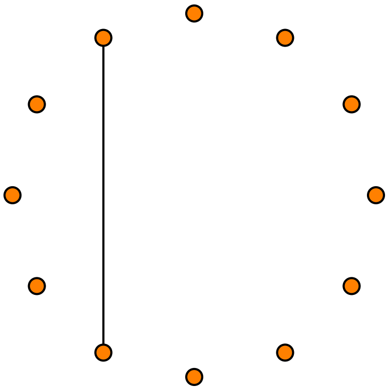

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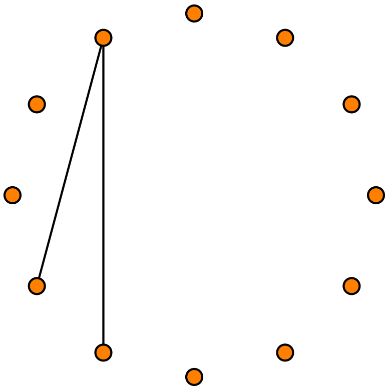

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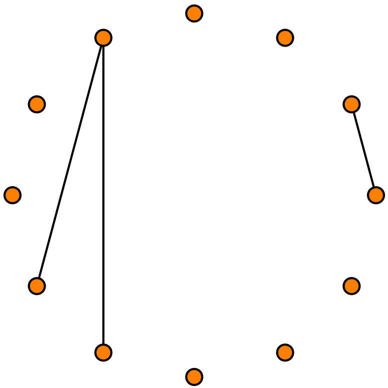

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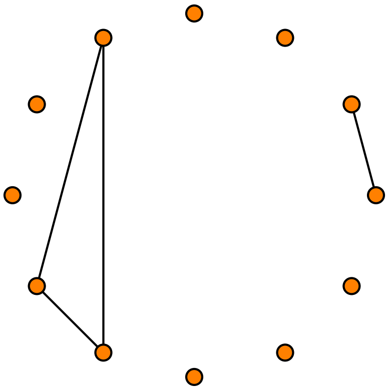

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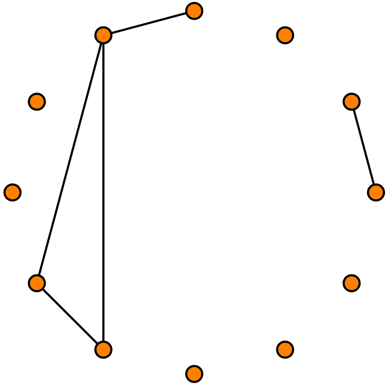

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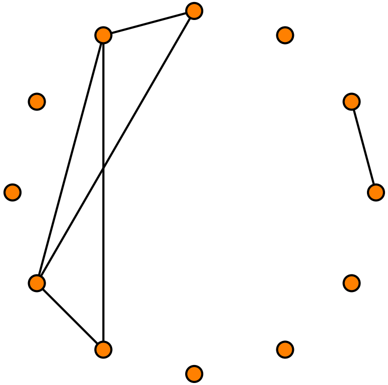

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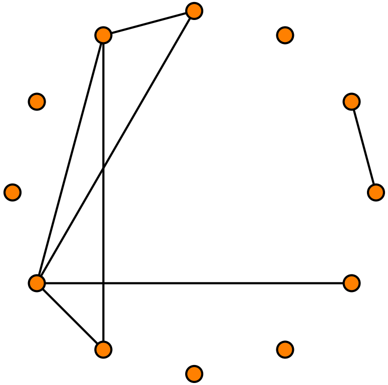

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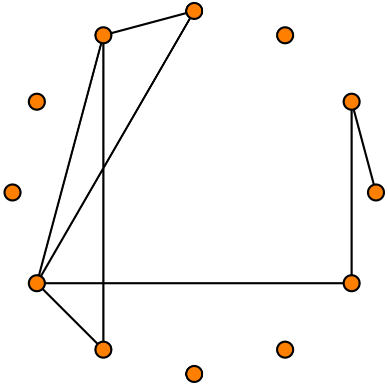

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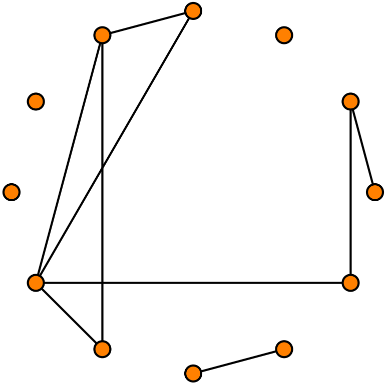

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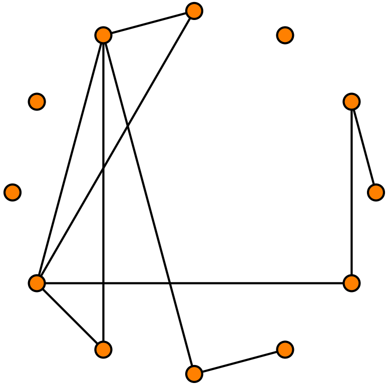

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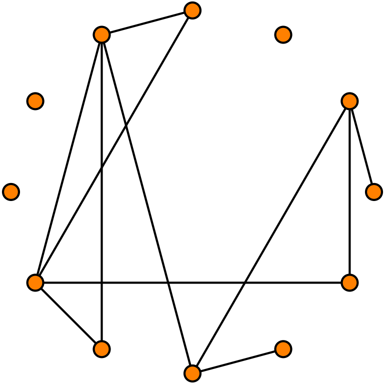

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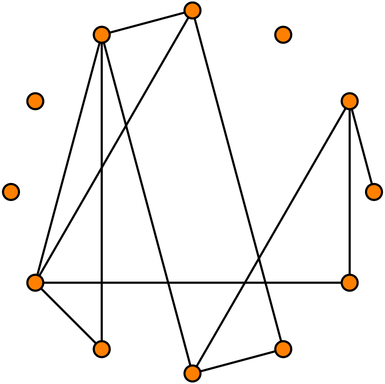

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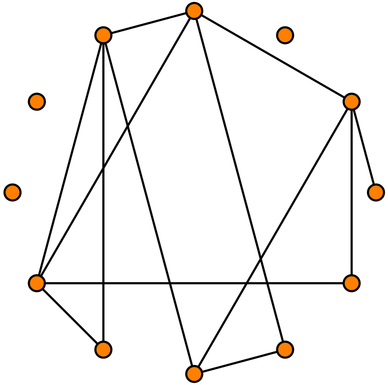

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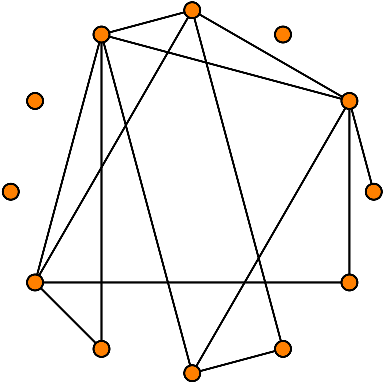

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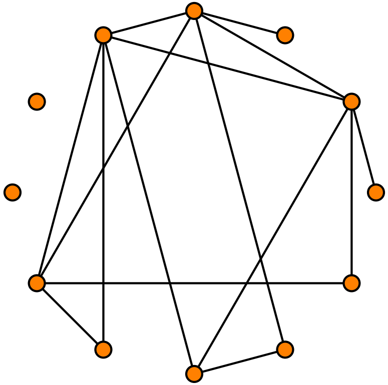

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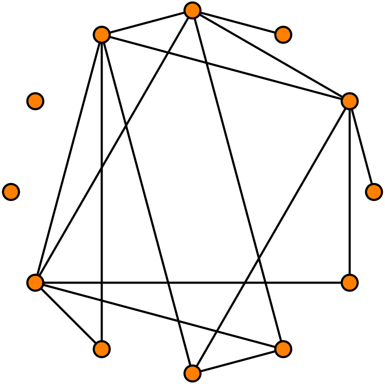

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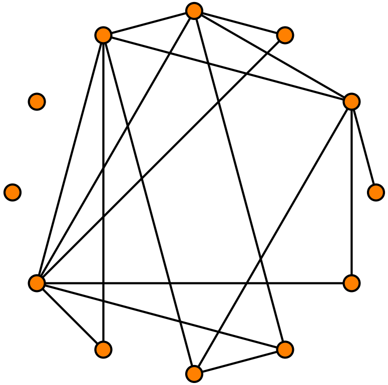

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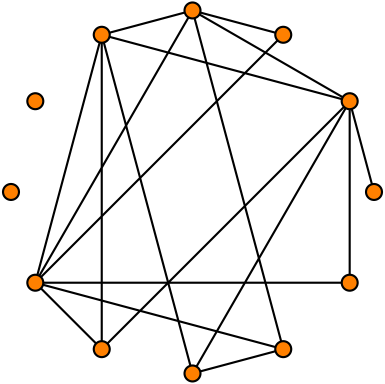

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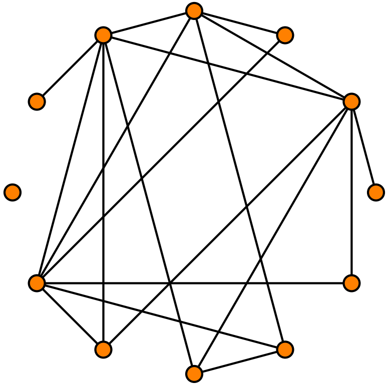

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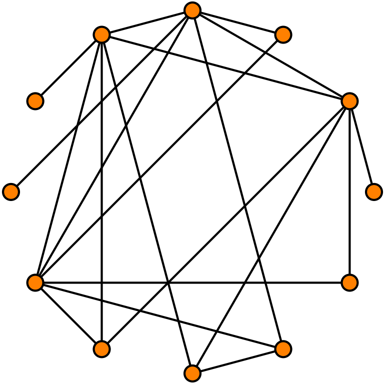

Connectivity of random graphs

Theorem (Erdős–Rényi, 1959)

Let $\epsilon > 0$ *be fixed and* $G \in G(n, p)$. *Then*

$$\mathbf{Pr}[G \textit{ is connected}] \to \begin{cases} 1 & : p \geq (1+\epsilon)\log n/n \\ 0 & : p \leq (1-\epsilon)\log n/n \end{cases}$$

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They actually proved a slightly sharper result.

Connectivity of random graphs

Theorem (Erdős–Rényi, 1959)

If

$$p = \frac{\log n + c}{n},$$

where $c \in \mathbb{R}$ *is constant then* $\widetilde{\beta}_0$ *is Poisson distributed and in particular*

$$\mathbf{Pr}[G \textit{ is connected}] \to e^{-e^{-c}}$$

as $n \to \infty$.

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The expected number of isolated vertices V is

$$\mathbb{E}[V] = n(1-p)^{n-1},$$

and since

$$1 - p \approx e^{-p},$$

when $p = \log n / n$, we have $\mathbb{E}[V] \to 1$.

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The next step is to show that the expected total number of components of order i, with $2 \leq i \leq n/2$ tends to 0 as $n \to \infty$.

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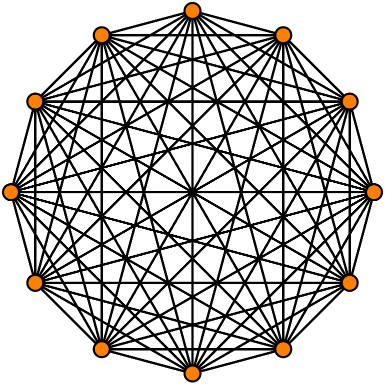
So the ultimate obstruction to connectivity is isolated vertices.

Random 2-complexes

Random 2-complexes

Linial and Meshulam defined $Y(n, p)$ to be the probability space of 2-dimensional simplicial complexes with vertex set $[n]$, edge set $\binom{[n]}{2}$, and such that each 2-face has probability p.

Random 2-complexes



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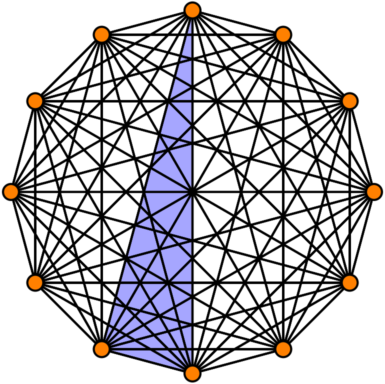

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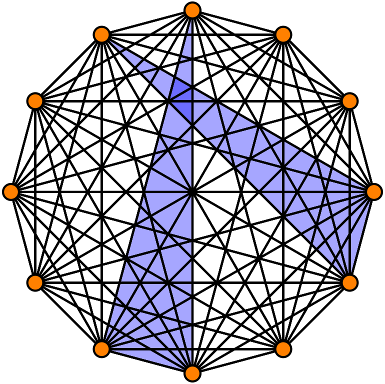

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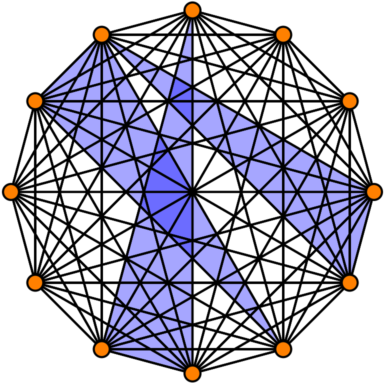

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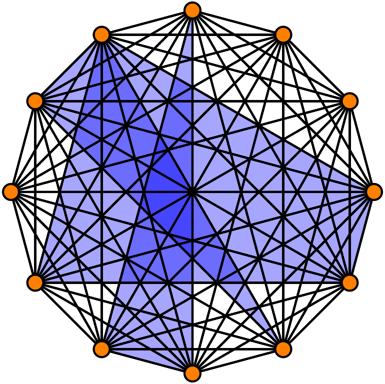

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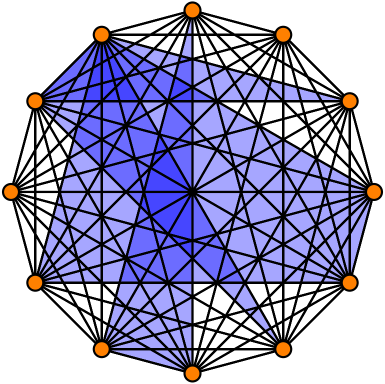

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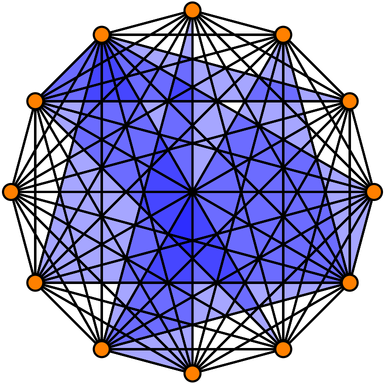

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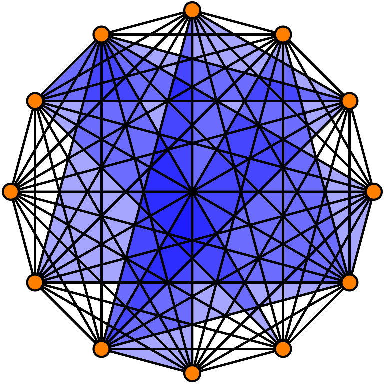

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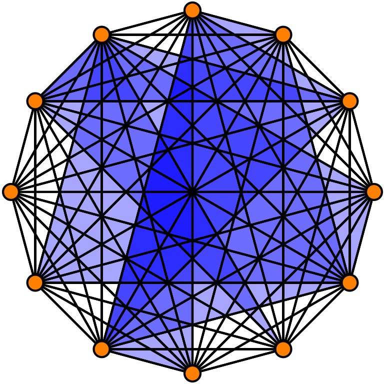

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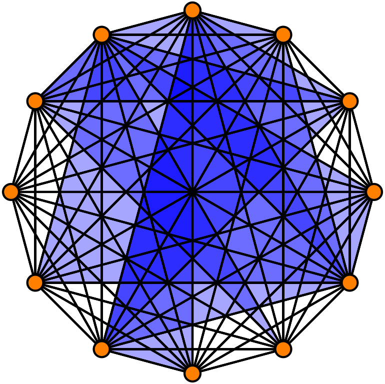

Random 2-complexes

Theorem

(Linial–Meshulam, 2006) Let $\epsilon > 0$ *be fixed and* $Y \in Y(n,p)$*. Then*

$$\mathbf{Pr}[H^1(Y, \mathbb{Z}/2) = 0] \to \begin{cases} 1 & : p \geq \frac{(2+\epsilon)\log n}{n} \\ 0 & : p \leq \frac{(2-\epsilon)\log n}{n} \end{cases}$$

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The ultimate obstruction to vanishing cohomology is isolated edges.

Random 2-complexes — results

The threshold for simple connectivity is much larger.

Theorem

(Babson–Hoffman–K., 2011) Let $\epsilon > 0$ *be fixed and* $Y \in Y(n,p)$. *Then*

$$\mathbf{Pr}[\pi_1(Y) = 0] \to \begin{cases} 1 & : p \gg \frac{\sqrt{n}}{n} \\ 0 & : p \ll \frac{\sqrt{n}}{n} \end{cases}$$

Random 2-complexes — results

Theorem

(Hoffman–K.–Paquette, 2012) Let $\epsilon > 0$ *be fixed and* $Y \in Y(n, p)$*. Then*

$$\mathbf{Pr}[\pi_1(Y) \textit{ has property (T)}] \to \begin{cases} 1 & : p \geq \frac{(2+\epsilon)\log n}{n} \\ 0 & : p \leq \frac{(2-\epsilon)\log n}{n} \end{cases}$$

Random 2-complexes — tools

Theorem

(Żuk) If Δ is a finite, connected, pure 2-dimensional simplicial complex, such that for every vertex v, the link $lk_{\Delta}(v)$ is connected and has spectral gap of normalized Laplacian satisfying $\lambda_2[lk_{\Delta}(v)] > 1/2$, then $\pi_1(\Delta)$ has property (T).

Random 2-complexes — tools

Theorem

(Hoffman – K. – Paquette, 2012) Fix $k \geq 0$ *and* $\epsilon > 0$, *and let* $G \in G(n,p)$. *Let* $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$ *be the eigenvalues of the normalized Laplacian of* G. *There is a constant* $C = C(k)$ *so that when*

$$p \geq \frac{(k+1)\log n + C\sqrt{\log n}\log\log n}{n}$$

is satisfied, then

$$\lambda_2 > 1 - \epsilon,$$

with probability at least $1 - o(n^{-k})$.

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Let $X \in X(n, p)$ be the clique complex (or flag complex) of $G \in G(n, p)$, i.e. the maximal simplicial complex compatible with G.

Note: every simplicial complex is homeomorphic to a flag complex, e.g. by barycentric subdivision, so $X(n, p)$ puts a measure on a wide range of topologies.

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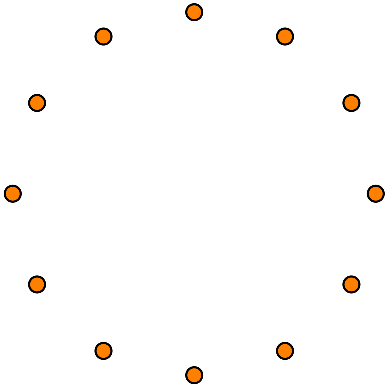

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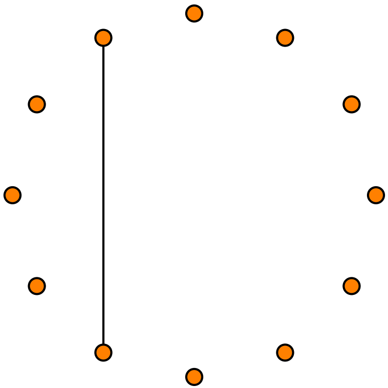

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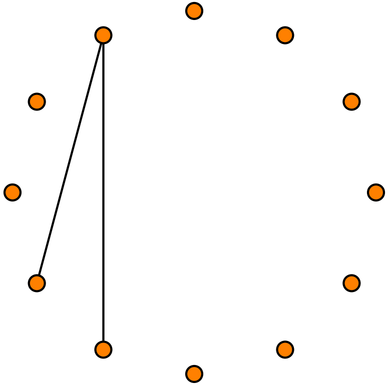

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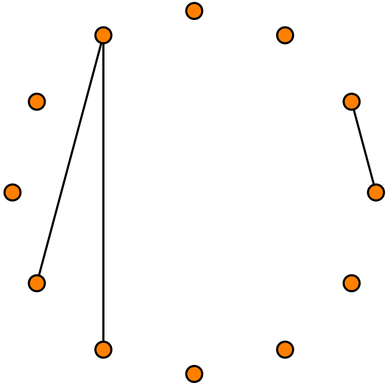

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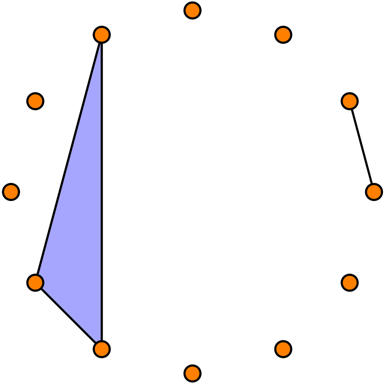

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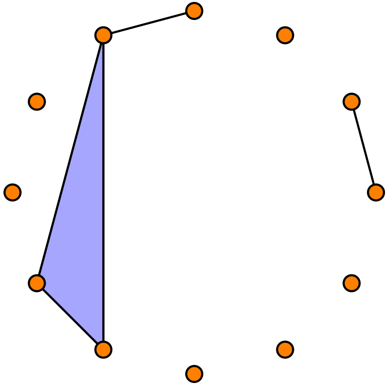

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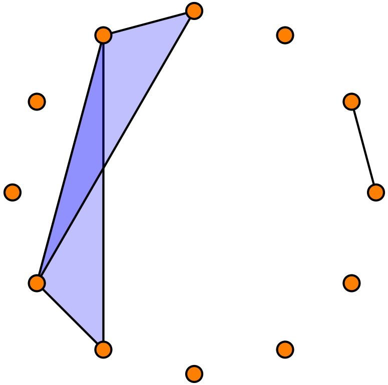

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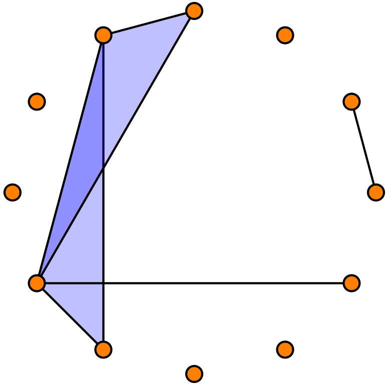

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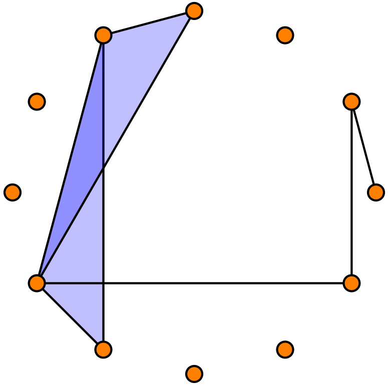

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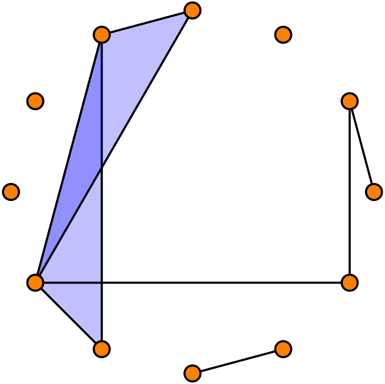

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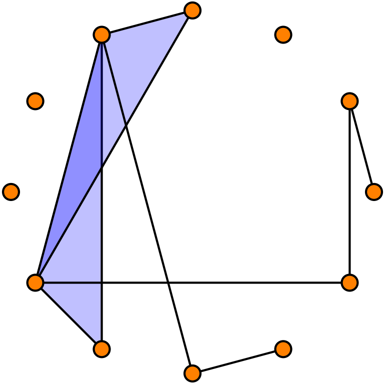

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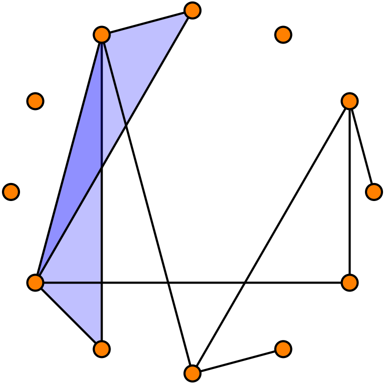

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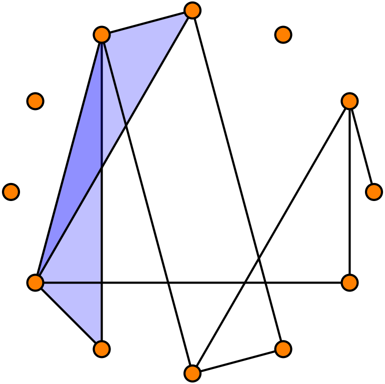

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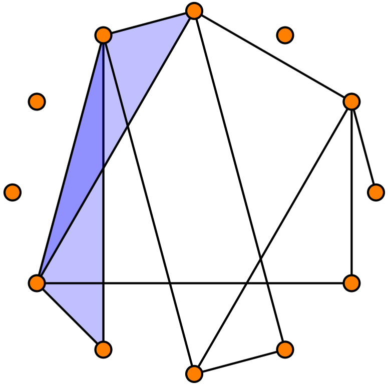

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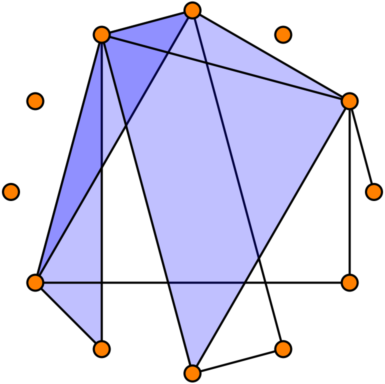

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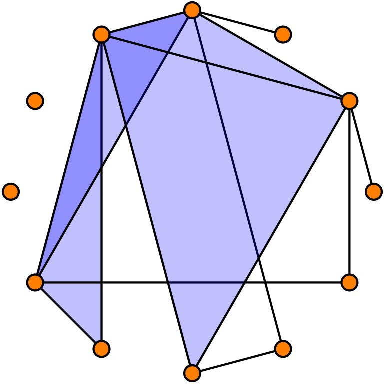

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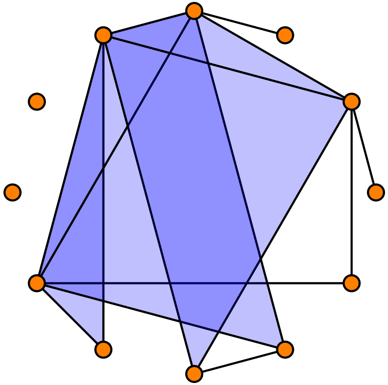
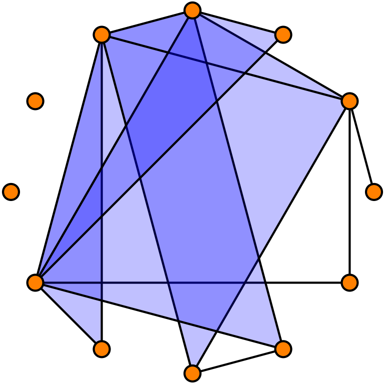
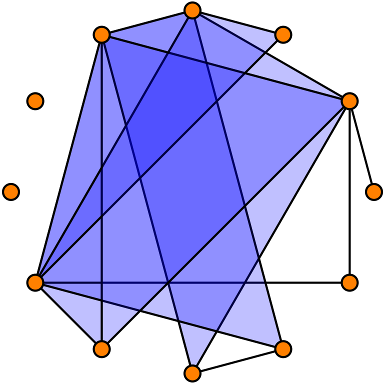


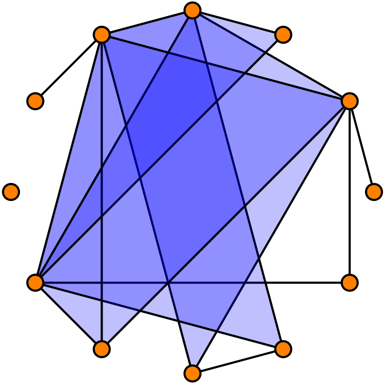
Random flag complexes



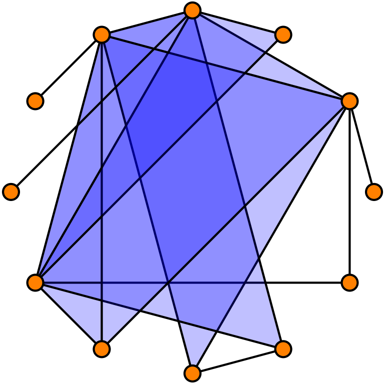
Random flag complexes



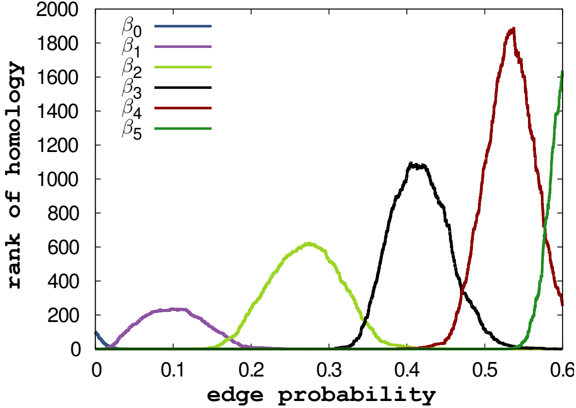
Random flag complexes



Random flag complexes



Random flag complexes



Random flag complexes on $n = 100$ vertices, with $0 \leq p \leq 0.6$. (Image courtesy of Afra Zomorodian.)

Random flag complexes — old results

Theorem

(K., 2007)

1. *If*

$$p \ll \frac{1}{n^{1/k}}$$

then

$$\mathbf{Pr}[H_k(X) = 0] \to 1$$

as $n \to \infty$.

Random flag complexes — old results

Theorem

(K., 2007)

1. *If*
$$p \ll \frac{1}{n^{1/k}}$$
then
$$\mathbf{Pr}[H_k(X) = 0] \to 1$$
as $n \to \infty$.

2. *Also, if*
$$p \gg \frac{1}{n^{1/(2k+1)}}$$
then
$$\mathbf{Pr}[H_k(X) = 0] \to 1$$
as $n \to \infty$.

Random flag complexes — old results

Theorem

(K., 2007) If

$$\frac{1}{n^{1/k}} \ll p \ll \frac{1}{n^{1/(k+1)}}$$

then

$$\mathbf{Pr}[H_k(X) = 0] \to 0$$

as $n \to \infty$.

Random flag complexes — old results

In fact, much more can be said about the size of homology in this regime. The limiting expectation has a nice formula.

Random flag complexes — old results

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Theorem

(K., 2007) If

$$\frac{1}{n^{1/k}} \ll p \ll \frac{1}{n^{1/(k+1)}}$$

then

$$\frac{\mathbb{E}[\beta_k]}{\binom{n}{k+1} p^{\binom{k+1}{2}}} \to 1$$

as $n \to \infty$.

Random flag complexes — old results

Moreover, the kth Betti number satisfies a central limit theorem.

Random flag complexes — old results

Moreover, the kth Betti number satisfies a central limit theorem.

Theorem

(K.–Meckes, 2010) If

$$\frac{1}{n^{1/k}} \ll p \ll \frac{1}{n^{1/(k+1)}}$$

then

$$\frac{\beta_k - \mathbb{E}[\beta_k]}{\sqrt{Var\left[\beta_k\right]}} \to \mathcal{N}(0,1)$$

as $n \to \infty$.

Random flag complexes — new results

Random flag complexes — new results

Theorem

(K., 2012) Let $0 < \epsilon < 1$ *and* k *be fixed and* $X \in X(n,p)$*. Then*

$$\mathbf{Pr}[H^k(Y,\mathbb{Q}) = 0] \to \begin{cases} 1 & : p \geq \left(\frac{(k/2+1+\epsilon)\log n}{n}\right)^{1/(k+1)} \\ 0 & : \frac{1}{n^{1/k}} \ll p \leq \left(\frac{(k/2+1-\epsilon)\log n}{n}\right)^{1/(k+1)} \end{cases}$$

Random flag complexes — tools

Theorem

(Garland, 1973, Ballman–Świątkowski, 1997) If Δ is a pure k-dimensional simplicial complex, such that the link $lk_\Delta(\sigma)$ of every $(k-2)$-face σ is connected and has spectral gap satisfying

$$\lambda_2[lk_\Delta(\sigma)] > 1 - 1/k,$$

then $H^{k-1}(\Delta, \mathbb{Q}) = 0$.

Random flag complexes — results

Applying universal coefficients for homology and cohomology we have the following.

Corollary

Fix $d \geq 0$, and let $X \in X(n,p)$ be a random flag complex, where

$$\frac{1}{n^{2/d}} \ll p \ll \frac{1}{n^{2/(d+1)}}.$$

Then w.h.p. X is d-dimensional and $\widetilde{H}_i(X, \mathbb{Q}) = 0$ unless $i = \lfloor d/2 \rfloor$.

Open problems

It might be possible to slightly sharpen our main result, and also to prove a Poisson approximation theorem.

Open problems

Conjecture

If

$$p = \left(\frac{(k/2+1)\log n + (k/2)\log\log n + c}{n} \right)^{1/(k+1)},$$

where $c \in \mathbb{R}$ is constant, then the dimension of kth cohomology β^k approaches a Poisson distribution with mean

$$\mu = \frac{(k/2+1)^{k/2}}{(k+1)!} e^{-c}.$$

In particular,

$$\mathbf{Pr}[H^k(X,\mathbb{Q}) = 0] \to \exp\left[- \frac{(k/2+1)^{k/2}}{(k+1)!} e^{-c} \right],$$

as $n \to \infty$.

Open problems

This conjecture is equivalent to showing that in this regime, the cohomology is generated by characteristic functions on maximal k-faces.

Open problems

How to handle torsion in homology of random complexes?

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There exists a 2-dimensional $\mathbb{Q}$-acyclic simplicial complex S on 31 vertices with

$$|H_1(S, \mathbb{Z})| = 736712186612810774591.$$

Open problems

How to handle torsion in homology of random complexes?

There exists a 2-dimensional $\mathbb{Q}$-acyclic simplicial complex S on 31 vertices with

$$|H_1(S, \mathbb{Z})| = 736712186612810774591.$$

Gil Kalai showed that, on average, $\mathbb{Q}$-acyclic complexes have enormous torsion in homology.

Open problems

Nevertheless, I conjecture the following.

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Conjecture

If $X \in X(n,p)$ is a random flag complex with

$$\frac{1}{n^{2/d}} \ll p \ll \frac{1}{n^{2/(d+1)}},$$

and if $d \geq 6$, then w.h.p. X is homotopy equivalent to a wedge of $\lfloor d/2 \rfloor$-spheres.

Open problems

By uniqueness of Moore spaces, this is equivalent to showing that $H_*(X)$ is torsion-free.

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