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Note

The neighborhood complex of a random graph

Matthew Kahle

Department of Mathematics, University of Washington, USA

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Abstract

For a graph G, the neighborhood complex $\mathcal{N}[G]$ is the simplicial complex having all subsets of vertices with a common neighbor as its faces. It is a well-known result of Lovász that if $\|\mathcal{N}[G]\|$ is k-connected, then the chromatic number of G is at least $k+3$.

We prove that the connectivity of the neighborhood complex of a random graph is tightly concentrated, almost always between $1/2$ and $2/3$ of the expected clique number. We also show that the number of dimensions of nontrivial homology is almost always small, $O(\log d)$, compared to the expected dimension d of the complex itself.

Keywords: Random graphs; Graph coloring; Morphism complexes

1. Introduction

In 1978, László Lovász proved Kneser's conjecture [11], that if the n-subsets of a $(2n+k)$-set are partitioned into $k+1$ families, at least one family contains a disjoint pair. He restated the problem graph theoretically and then proved a more general theorem about graph coloring.

All our graphs will be simple undirected graphs, with no loops or multiple edges. For a graph G, a *k-coloring* is a function $f: V(G) \to \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever $\{x, y\} \in E(G)$. The *chromatic number* $\chi(G)$ is the minimum k such that G admits a k-coloring. Define the Kneser graph $KG(n,k)$ to have all n-subsets of a $(2n+k)$-set as its vertices, with edges between disjoint pairs. The Kneser conjecture is equivalent to the claim that $\chi(KG(n,k)) \geqslant k+2$.

E-mail address: kahle@math.washington.edu.

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There are several simplicial complexes naturally associated with a graph G. The clique complex $X(G)$ is the simplicial complex $X(G)$ on vertex set $V(G)$ whose simplices are all complete subgraphs of G. In another article [10], we study the clique complex of a random graph. The *neighborhood complex* $\mathcal{N}[G]$ is the simplicial complex on $V(G)$ which has all subsets of $V(G)$ with a common neighbor for its faces. For example, the neighborhood complex of the complete graph K_n has all proper subsets of the vertices for its faces, so it is the boundary of an $(n-1)$-dimensional simplex. Its geometric realization $\|\mathcal{N}[K_n]\|$ is homeomorphic to an $(n-2)$-dimensional sphere $\mathbb{S}^{n-2}$.

A topological space X is said to be *k-connected* if every map from a sphere $\mathbb{S}^n \to X$ extends to a map from the ball $\mathbb{B}^{n+1} \to X$ for $n = 0, 1, \ldots, k$.

Theorem 1.1 *(Lovász). If $\|\mathcal{N}[G]\|$ is k-connected, then $\chi(G) \geqslant k+3$.*

In the case of the Kneser graphs this lower bound is tight, matching an easy upper bound to give an exact answer. It seems natural to ask how good the bound is for "typical" graphs. For our purposes, a typical graph is the random graph $G(n, p)$ [4].

The random graph $G(n, p)$ is the probability space of all graphs on a vertex set of size n with each edge inserted independently with probability p. Frequently, one considers p to be a function of n and asks whether the graph is likely to have some property $\mathcal{P}$ as $n \to \infty$. We say that $G(n, p) \in \mathcal{P}$ *almost always* (*a.a.*) if $\mathrm{Prob}[G(n, p) \in \mathcal{P}] \to 1$ as $n \to \infty$. The main goal of this article is to understand some of the most basic topological features of $\|\mathcal{N}[G(n, p)]\|$.

2. Statement of results

Let $p = p(n)$ be a monotone function of n, and let i, j, k, and l be integer valued monotone functions of n. In the asymptotic notation that follows, $n \to \infty$ is the free variable. Homology is understood to be reduced with coefficients in $\mathbb{Z}$ throughout.

Theorem 2.1. *If $\binom{n}{i}(1-p^i)^{n-i} = o(1)$ then $\mathcal{N}[G(n, p)]$ is a.a. $(i-2)$-connected.*

Theorem 2.2. *If $\binom{n}{j}\binom{n}{k}p^{jk} = o(1)$ then $\mathcal{N}[G(n, p)]$ a.a. strong deformation retracts to a simplicial complex of dimension at most $j+k-3$.*

In proving Theorem 2.2, we make use of the following lemma, which is implicit in [7], but which we include here with proof for the sake of completeness.

Lemma 2.3. *If H is any graph not containing a complete bipartite subgraph $K_{a,b}$ then $\|\mathcal{N}[H]\|$ strong deformation retracts to a complex of dimension at most $a+b-3$.*

A d-connected complex has trivial homology through dimension d by the Hurewicz theorem. Also, $\widetilde{H}_k(\Delta) = 0$ whenever k is greater than the dimension of Δ, and strong deformation retracts are homotopy equivalences and preserve homology. Hence Theorems 2.1 and 2.2 bound the possible dimensions of nontrivial homology from below and above. We give special cases of the theorems as corollaries. First fix $p = 1/2$, where $G(n, p)$ is the uniform distribution on all graphs on vertex set $[n]$.

Corollary 2.4. *If $p = 1/2$ and $\epsilon > 0$ then a.a. $\widetilde{H}_l(\|\mathcal{N}[G(n, p)]\|) = 0$ for $l \leqslant (1-\epsilon)\log_2 n$ and $l \geqslant (4+\epsilon)\log_2 n$.*

Note for comparison that the dimension of the neighborhood complex is one less than the maximum vertex degree, so when $p = 1/2$ we expect it to be slightly more than $n/2$. Next, fix l and check that $\widetilde{H}_l$ is trivial outside a certain range of p.

Corollary 2.5. *Let $p = n^\alpha$ with $\alpha \in [-2, 0]$. If $\alpha > \frac{-1}{l+2}$ then a.a. $\widetilde{H}_l(\|\mathcal{N}[G(n,p)]\|) = 0$. For l even, if $\alpha < \frac{-4}{l+2}$ then a.a. $\widetilde{H}_l(\|\mathcal{N}[G(n,p)]\|) = 0$. For l odd, if $\alpha < \frac{-4(l+2)}{(l+1)(l+3)}$ then a.a. $\widetilde{H}_l(\|\mathcal{N}[G(n,p)]\|) = 0$.*

For a partial converse to Corollaries 2.4 and 2.5, we exhibit explicit nontrivial homology classes by retracting onto random spheres. Recall that a *clique* of order n is a complete subgraph on n vertices.

Definition 2.6. Let the graph X_n have vertex set $\{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$, such that $\{u_1, u_2, \ldots, u_n\}$ spans a clique, and u_i is adjacent to v_j whenever $i \neq j$.

Theorem 2.7. *If H is any graph containing a maximal clique of order n that cannot be extended to an X_n subgraph, then $\|\mathcal{N}[H]\|$ retracts onto a sphere $\mathbb{S}^{n-2}$.*

Corollary 2.8. *If $p = 1/2$, $\epsilon > 0$, and $(4/3 + \epsilon)\log_2 n < k < (2 - \epsilon)\log_2 n$, then a.a. $\widetilde{H}_k(\|\mathcal{N}[G(n,p)]\|) \neq 0$.*

Corollary 2.9. *Let $p = n^\alpha$ with $\frac{-2}{k+1} < \alpha < \frac{-4}{3(k+1)}$, then a.a. $\widetilde{H}_k(\|\mathcal{N}[G(n,p)]\|) \neq 0$.*

3. Proofs

We first prove that if $\binom{n}{i}(1 - p^i)^{n-i} = o(1)$ then $\mathcal{N}[G(n,p)]$ is a.a. $(i-2)$-connected.

Proof of Theorem 2.1. A simplicial complex is *i-neighborly* if every i vertices span a face. By simplicial approximation, if a complex is i-neighborly then it is $(i-2)$-connected. The probability that a given set of i vertices in $G(n,p)$ has no neighbor is $(1 - p^i)^{n-i}$. Then the total probability that any set of i vertices does not have a neighbor is bounded above by $\binom{n}{i}(1 - p^i)^{n-i} = o(1)$. So a.a. every such set has some common neighbor, hence spans a face in the neighborhood complex. So $\mathcal{N}[G(n,p)]$ is a.a. i-neighborly and $(i-2)$-connected. $\square$

Next we prove that if H is any graph not containing a complete bipartite subgraph $K_{a,b}$ then $\|\mathcal{N}[H]\|$ is homotopy equivalent to a complex of dimension at most $a + b - 3$.

Proof of Lemma 2.3. For a poset Q, the *order complex* $\Delta(Q)$ is the simplicial complex of all chains in Q. For a simplicial complex S, let $P(S)$ denote its face poset. To avoid proliferation of notation, we denote the geometric realization of the order complex of a poset Q by $\|Q\|$ rather than $\|\Delta(Q)\|$.

For a vertex x of H, let $\Gamma(x)$ denote the set of common neighbors of x. Similarly, for any face in the neighborhood complex $X \in \mathcal{N}[H]$, let $\Gamma(X) = \bigcap_{x \in X} \Gamma(x)$. (This map is used in Lovász's paper [11].) Note that Γ is an order reversing self-map of $P(\mathcal{N}[H])$, abbreviated for the rest of this proof by P. So we can define an order preserving poset map $v : P \to P$ by $v(X) = \Gamma(\Gamma(X))$. It is also easy to check that $\Gamma^3 = \Gamma$, so $\Gamma^4 = \Gamma^2$, and $v^2 = v$. Since $v(X) \supseteq X$ for every X, a standard theorem in combinatorial homotopy theory [5] gives that $\|v(P)\|$ is a strong deformation retract of $\|P\|$.

An $(m+1)$-dimensional face in $v(P)$ is a chain of faces in $\mathcal{N}[H]$, $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_{m+2}$. Set $Y_i = \Gamma(X_i)$ and we have $Y_1 \supsetneq Y_2 \supsetneq \cdots \supsetneq Y_{m+2}$. (The inclusions $Y_i \supsetneq Y_{i+1}$ are strict since $\Gamma^3 = \Gamma$.)

Suppose $a + b - 3 \leqslant m$. Since the inclusions $X_i \subsetneq X_{i+1}$ are strict and X_1 is nonempty, X_a contains at least a vertices. Similarly, Y_a contains at least $m + 3 - a \geqslant b$ vertices. But $Y_p = \Gamma(X_p)$, so X_a and Y_b span a complete bipartite subgraph $K_{a,b}$. Then if the dimension of $v(P)$ is at least $m+1$, H contains $K_{a,b}$ subgraphs for every a and b such that $a + b - 3 \leqslant m$, which is the claim. □

Now we apply Lemma 2.3 to check that if $\binom{n}{j}\binom{n}{k}p^{jk} = o(1)$ then $\mathcal{N}[G(n,p)]$ a.a. strong deformation retracts to a simplicial complex of dimension at most $j + k - 3$.

Proof of Theorem 2.2. Let U and V be vertex subsets of $G(n,p)$ of order j and k, respectively. The probability that they span a complete bipartite graph with parts U and V is p^{jk}. So the total probability that there are any $K_{j,k}$ subgraphs is bounded above by $\binom{n}{j}\binom{n}{k}p^{jk} = o(1)$. There are a.a. no such subgraphs, so the claim follows by Lemma 2.3. □

Checking Corollaries 2.4 and 2.5 is now a straightforward computation.

Proof of Corollary 2.4. Let $p = 1/2$ and $\epsilon > 0$. If $l \leqslant (1-\epsilon)\log_2 n$, then

$$\begin{aligned}\binom{n}{l}\left(1-p^l\right)^{n-l} &= \binom{n}{l}\left(1-(1/2)^l\right)^{n-l} \leqslant n^l e^{-(1/2)^l(n-l)} \leqslant n^{\log_2 n} e^{-n^{-1+\epsilon}(n-\log_2 n)} \\ &= \exp\left(\log n \log_2 n - n^{\epsilon} + n^{-1+\epsilon}\log_2 n\right) = o(1).\end{aligned}$$

Then Theorem 2.1 gives that $\mathcal{N}[G(n,p)]$ is a.a. $(l-2)$-connected. Since ϵ does not appear anywhere in the conclusion of the theorem, we can replace it by a slightly smaller ϵ and for large enough n, $l + 2 \leqslant (1-\epsilon)\log_2 n$ and this gives that $\mathcal{N}[G(n,p)]$ is a.a. l-connected.

On the other hand, suppose $l \geqslant (4+\epsilon)\log_2 n$ and let $j = k = \lceil l/2 \rceil$.

$$\begin{aligned}\binom{n}{j}\binom{n}{k}\left(\frac{1}{2}\right)^{jk} &\leqslant n^{2(2+\epsilon/2)\log_2 n}\left(\frac{1}{2}\right)^{(2+\epsilon/2)^2(\log_2 n)^2} = n^{2(2+\epsilon/2)\log_2 n-(2+\epsilon/2)^2\log_2 n} \\ &= n^{-(\epsilon+\epsilon^2/4)\log_2 n} = o(1).\end{aligned}$$

Then Theorem 2.2 gives that $\mathcal{N}[G(n,p)]$ a.a. strong deformation retracts to a complex of dimension at most $2j - 3 \leqslant l - 1$. □

Proof of Corollary 2.5. Let $p = n^{\alpha}$ and suppose first that $\alpha > \frac{-1}{l+2}$,

$$\begin{aligned}\binom{n}{l+2}\left(1-p^{l+2}\right)^{n-l-2} &\leqslant n^{l+2}e^{-n^{\alpha(l+2)}(n-l-2)} \\ &\leqslant \exp\left[(l+2)\log n - n^{1+\alpha(l+2)} + n^{\alpha(l+2)}(l+2)\right] = o(1),\end{aligned}$$

since l is constant and $1 + \alpha(l+2) > 0$. Then Theorem 2.1 gives that $\mathcal{N}[G(n,p)]$ is a.a. l-connected. So a.a. $\widetilde{H}_l(\|\mathcal{N}[G(n,p)]\|) = 0$.

Now suppose l is even and $\alpha < \frac{-4}{l+2}$. Set $j = \frac{l+2}{2}$,

$$\binom{n}{j}^2 \left(n^\alpha\right)^{j^2} \leqslant n^{l+2} n^{\alpha(l+2)^2/4} = n^{(l+2)(4+\alpha(l+2))/4} = o(1),$$

since $4 + \alpha(l+2) < 0$. So Theorem 2.2 gives that $\mathcal{N}[G(n, p)]$ a.a. strong deformation retracts to a complex of dimension at most $2j - 3 = l - 1$.

Similarly, suppose l is odd and $\alpha < \frac{-4(l+2)}{(l+1)(l+3)}$. Set $j = \frac{l+1}{2}$,

$$\binom{n}{j}\binom{n}{j+1}\left(n^\alpha\right)^{j(j+1)} \leqslant n^{2j+1} n^{\alpha j(j+1)} \leqslant n^{l+2} n^{\alpha \frac{l+1}{2}\frac{l+3}{2}} = o(1).$$

Then Theorem 2.2 gives that $\mathcal{N}[G(n, p)]$ a.a. strong deformation retracts to a complex of dimension at most $2j - 2 = l - 1$. In both the even and odd cases $\widetilde{H}_l(\|\mathcal{N}[G(n, p)]\|) = 0$. □

Recall that the graph X_n has vertex set $\{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$, such that $\{u_1, u_2, \ldots, u_n\}$ spans a clique, $\{v_1, v_2, \ldots, v_n\}$ is an independent set, and u_i is adjacent to v_j whenever $i \neq j$. We show now that if H is any graph containing a maximal clique $\{u_1, u_2, \ldots, u_n\}$ that is not contained in an X_n subgraph, then $\|\mathcal{N}[H]\|$ retracts onto a sphere $\mathbb{S}^{n-2}$.

Proof of Theorem 2.7. Suppose H contains a clique $X = \{u_1, u_2, \ldots, u_n\}$ that is not contained in any larger clique or X_n subgraph. The induced subcomplex of $\mathcal{N}[H]$ on X is a topological sphere $\mathbb{S}^{n-2}$, since X itself is not a face by assumption of maximality of the clique. For $i = 1, 2, \ldots, n$, define functions $r_i : V(H) \to V(H)$ by $r_i(x) = x$ for $x \in X$ and $r_i(x) = u_i$ otherwise.

The only possible obstruction to r_i extending to a simplicial map $\widetilde{r_i} : \mathcal{N}[H] \to \mathcal{N}[H]$, is an $(n-1)$-dimensional face getting mapped onto X. This happens only if for some vertex u_i^*, $X \cup \{u_i^*\} - u_i$ has a common neighbor v_i. Note that u_i is not adjacent to v_i since then $X \cup \{v_i\}$ would be an extension of X to a larger clique. If none of the candidate maps $r_1, \ldots, r_n$ extends to a simplicial map, then the v_i are clearly distinct, since the u_i are distinct and u_i is adjacent to v_j if and only if $i \neq j$. But this yields an X_n subgraph containing X. Otherwise $\|\mathcal{N}[H]\|$ retracts onto $\|\mathcal{N}[X]\| = \mathbb{S}^{n-2}$ as claimed, via one of these maps. □

Proof of Corollary 2.8. Let $p = 1/2$ and $\epsilon > 0$. It is well known that $G(n, p)$ a.a. contains maximal cliques of every order k with $(1+\epsilon)\log_2 n < k < (2-\epsilon)\log_2 n$ [4]. We need only check that there are a.a. no X_k subgraphs when $k > (4/3 + \epsilon)\log_2 n$. Note that X_k has $2k$ vertices and $3k(k-1)/2$ edges. Then the probability that $G(n, p)$ contains a copy of X_k is bounded above by

$$\begin{aligned}(2k)!\binom{n}{2k}\left(\frac{1}{2}\right)^{3k(k-1)/2} &\leqslant n^{2k}\left(\frac{1}{2}\right)^{3k(k-1)/2} \\ &\leqslant n^{(8/3+2\epsilon)\log_2 n} n^{(-3/2)(4/3+\epsilon)((4/3+\epsilon)\log_2 n - 1)} \\ &= n^{(8/3+2\epsilon)\log_2 n} n^{-(8/3+4\epsilon+3\epsilon^2/2)\log_2 n + 2 + 3\epsilon/2} \\ &= n^{-(2+3\epsilon/2)(\epsilon \log_2 n - 1)} = o(1). \qquad \square\end{aligned}$$

Proof of Corollary 2.9. Define the *density* of a graph with v vertices and e edges to be $\lambda = e/v$. We say a graph is *strictly balanced* if the density of the graph itself is strictly greater than the density of any of its subgraphs.

Let H be any strictly balanced graph of density λ. It is classical that $p = n^{-1/\lambda}$ is a sharp threshold for $G(n, p)$ containing H as a subgraph [4]. In particular, if $p = n^\alpha$ and $\alpha > -1/\lambda$ then $G(n, p)$ a.a. contains H as a subgraph, and if $\alpha < -1/\lambda$ then $G(n, p)$ a.a. does not contain H.

Since K_k and X_k are both strictly balanced we may apply this result twice. The density of K_k is $(k-1)/2$, and the density of X_k is $3(k-1)/4$. So if $p = n^\alpha$ with $\frac{-2}{k+1} < \alpha < \frac{-4}{3(k+1)}$, then $G(n, p)$ a.a. contains K_{k+2} but not X_{k+2} subgraphs. This implies that $\widetilde{H}_k(\|\mathcal{N}[G(n, p)]\|) \neq 0$ by Theorem 2.7 once we check the detail that at least one of these K_{k+2} subgraphs cannot be extended to a K_{k+3}. In fact a randomly chosen clique will do the job. The conditional probability that a given K_{k+2} extends to a K_{k+3} is easily seen to be bounded above by np^{k+2}, and $np^{k+2} = o(1)$ since $p = n^\alpha$ with $\alpha < \frac{-4}{3(k+1)}$. □

4. Topological bounds on chromatic number

The chromatic number $\chi(G(n, 1/2))$ is tightly concentrated around $n/\log_2 n$. For comparison, the clique number is almost always close to $2\log_2 n$. As a corollary of what we have shown here, the connectivity of the neighborhood complex, somewhere between $\log_2 n$ and $(4/3)\log_2 n$, is almost always less than the clique number.

Similar remarks hold for all monotone functions $p = p(n)$. The asymptotic picture that emerges is the following. The neighborhood complex strong deformation retracts to a complex of dimension d, which is $d/4$-connected, with nonvanishing homology between dimensions $d/3$ and $d/2$, where the clique number is $d/2$. We see why the connectedness of the neighborhood complex actually gives a worse bound on chromatic number than the clique number for random graphs; the maximal cliques themselves actually represent nontrivial homology classes.

Recent work of Eric Babson and Dmitry Kozlov [1–3] provides new examples of topological bounds on chromatic number. Briefly, to any pair of graphs G and H, one may associate a polyhedral complex $\mathrm{Hom}(G, H)$ of all generalized graph maps $G \mapsto H$. There has been a flurry of recent activity on these complexes [6–8,13]. A nice introduction to the subject can be found in [2].

A well-known fact in this area is that $\mathrm{Hom}(K_2, H)$ is homotopy equivalent to $\mathcal{N}[H]$. So Lovász's original result (Theorem 1.1) is equivalent to the statement that

$$\chi(H) \geqslant \text{connectivity}\big(\mathrm{Hom}(K_2, H)\big) + 3.$$

Call a topological space T, together with a free $\mathbb{Z}_2$-action, a *free $\mathbb{Z}_2$-space*. For example, spheres are free $\mathbb{Z}_2$-spaces, with respect to the antipodal map. Define the $\mathbb{Z}_2$-index of such a space by

$$\mathrm{ind}(T) := \min\big\{n \geqslant 0\text{: there exists a } \mathbb{Z}_2\text{-equivariant map } T \mapsto S^n\big\}.$$

For any such space T we have

$$\dim(T) \geqslant \mathrm{ind}(T) \geqslant 1 + \text{connectivity}(T).$$

The nontrivial automorphism of K_2 induces a free $\mathbb{Z}_2$-action on $\mathrm{Hom}(K_2, H)$, and Lovász's result can be strengthened to say that for any graph H,

$$\chi(H) \geqslant \mathrm{ind}\big(\mathrm{Hom}(K_2, H)\big) + 2.$$

However, we have shown in this article that $\mathrm{Hom}(K_2, G(n, p))$ is a.a. homotopy equivalent to a complex of small dimension (no more than twice the clique number). Then since dimension is an upper bound on index, this strengthening cannot improve the bound much for random graphs.

Babson and Kozlov showed [3] that for any odd cycle C_{2r+1} and graph H,

$$\chi(H) \geqslant \mathrm{ind}\big(\mathrm{Hom}(C_{2r+1}, H)\big) + 3.$$

However, Carsten Schultz showed that in fact

$$\operatorname{ind}\big(\operatorname{Hom}(C_{2r+1}, H)\big) \leqslant \operatorname{ind}\big(\operatorname{Hom}(K_2, H)\big) + 1,$$

and also that equality holds for large enough r [13]. So the Babson–Kozlov bounds are asymptotically identical to the Lovász bound.

It seems then that none of the well studied topological bounds on chromatic number can do much better than clique number for random graphs. However, there are still many intriguing open questions in this area. Call T a *homology test graph* if for every H,

$$\chi(H) \geqslant \operatorname{ind}\big(\operatorname{Hom}(T, H)\big) + \chi(T).$$

It would be interesting to know if for every fixed graph H, there exists a homology test graph T such that equality holds.

The strongest statement one might hope for in this direction is that for every pair of graphs there exists a homology test graph T, depending on the pair, that gives a tight bound on chromatic number simultaneously for both graphs. Then functoriality of the Hom-complexes would give a proof of the infamous Hedetnemi graph product conjecture, that for any pair of graphs,

$$\chi(H \times H') = \min\big(\chi(H), \chi(H')\big).$$

5. Random simplicial complexes

One justification for random graph theory is that it provides models for "typical" graphs. This can be made precise in a few ways. For example, $G(n, 1/2)$ is the uniform distribution on all graphs on vertex set $[n]$. Any property that $G(n, 1/2)$ a.a. has is a property of almost all graphs. Or for another example, the Szemerédi Regularity Lemma states that every graph is well approximated by random graphs.

Every neighborhood complex is homotopy equivalent to a free $\mathbb{Z}_2$-complex, via Lovász's map used in the Proof of Lemma 2.3. Up to homotopy, the converse also holds [6].

Theorem 5.1 *(Csorba). Given a finite simplicial complex Δ with a free $\mathbb{Z}_2$-action, there exists a graph G such that $\|\mathcal{N}[G]\|$ is homotopy equivalent to $\|\Delta\|$.*

So up to homotopy type, $\mathcal{N}[G(n, p)]$ is a probability space of all finite $\mathbb{Z}_2$-complexes as $n \to \infty$. Then the results here can be read as topological statements about "typical" antipodality spaces, at least to whatever extent that one believes that $\mathcal{N}[G(n, p)]$ is a natural measure.

Little seems to be written so far about random simplicial complexes. However, Nathan Linial and Roy Meshulam recently studied $H_1(\|Y(n, p)\|, \mathbb{Z}_2)$ for random 2-dimensional simplicial complexes $Y(n, p)$ [12]. Their definition of $Y(n, p)$ is a natural extension of the Erdős–Rényi random graph $G(n, p)$; $Y(n, p)$ has vertex set $[n]$ and edge set $\binom{[n]}{2}$, with each 2-face appearing independently with probability p. One advantage of the Linial–Meshulam model is that vanishing of homology is a monotone property. That is, once enough 2-faces have been added that $H_1(\|Y(n, p)\|, \mathbb{Z}_2)$ vanishes, adding more 2-faces cannot ever make it nonvanishing. Monotonicity in this case does not depend on the coefficients of homology, but only on the fact that the underlying graph is assumed to be complete. Simple connectivity is also a monotone property but it is still not known where the threshold function lies. However bounds are placed on this threshold in [9], and it is known in particular that vanishing of $\pi_1(Y(n, p))$ and $H_1(\|Y(n, p)\|, \mathbb{Z}_2)$ do not have the same threshold function.

Most properties of $G(n, p)$ that have been studied to date are monotone graph properties, in contrast to what we have studied in this article, where vanishing of homology is not monotone. But in this setting unimodality may be a natural substitute for monotonicity.

In another article [10], we study the *clique complex* of a random graph, which is the simplicial complex with all complete subgraphs of $G(n, p)$ as faces. The results are somewhat analogous to what we find here, although nontrivial homology seems much more tightly concentrated and to refine the picture we also study the expectation of the Betti numbers β_k. We conjecture that for both random neighborhood and clique complexes, for any fixed k and large enough n depending on k, the expectation $E[\beta_k]$ is a unimodal function of p.

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