NEW LOWER BOUNDS ON $\chi(\mathbb{R}^d)$ FOR $d = 8, \ldots, 12$

Matthew Kahle
Ohio State University
mkahle@math.osu.edu

Birra Taha
Cornell University
bit2002@med.cornell.edu

1 Introduction

We study the infinite graph whose vertices correspond to points in d-dimensional Euclidean space and where two vertices p, q are considered adjacent $p \sim q$ if $d(p,q) = 1$. We denote this graph by $\mathbb{R}^d$.

Recall that the chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed for a proper coloring, i.e. the smallest k so that there exists a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that whenever $p \sim q$, we have $f(p) \neq f(q)$.

Determining the chromatic number of the plane $\chi\left(\mathbb{R}^2\right)$ is a well known problem in combinatorics. For several decades the bounds have held fast at

$$4 \leq \chi\left(\mathbb{R}^2\right) \leq 7.$$

See Soifer's "The mathematical coloring book" [7] for an encyclopedic overview of this problem and its history.

The problem has also been studied in higher dimensions. See [7] and Kupavskii–Raigorodskii's paper [4]. With the aid of extensive calculations in the free and open-source software Sage [8], we obtain new lower bounds for $\chi(\mathbb{R}^d)$ in dimensions $d = 8 \ldots 12$. We verified all of the computations in Maple 17.

Our main result is the following.

Theorem 1.1. *We have that* $\chi\left(\mathbb{R}^8\right) \geq 19$*,* $\chi\left(\mathbb{R}^9\right) \geq 21$*,* $\chi\left(\mathbb{R}^{10}\right) \geq 26$*,* $\chi\left(\mathbb{R}^{11}\right) \geq 32$*, and* $\chi\left(\mathbb{R}^{12}\right) \geq 32$*.*

As far as we are aware, the best previously published lower bounds were, respectively:
$\chi\left(\mathbb{R}^8\right) \geq 16$ by Larman and Rogers in 1972 [5], and $\chi\left(\mathbb{R}^9\right) \geq 21$, $\chi\left(\mathbb{R}^{10}\right) \geq 23$, $\chi\left(\mathbb{R}^{11}\right) \geq 23$, and $\chi\left(\mathbb{R}^{12}\right) \geq 25$ by Kupavskii and Raigorodskii in 2009 [4].

The rest of the paper is organized as follows. We prove the bounds $\chi\left(\mathbb{R}^{10}\right) \geq 26$, $\chi\left(\mathbb{R}^{11}\right) \geq 32$, and $\chi\left(\mathbb{R}^{12}\right) \geq 32$ in Section 2; the bound $\chi\left(\mathbb{R}^9\right) \geq 21$ in Section 3; and the bound $\chi\left(\mathbb{R}^8\right) \geq 19$ in Section 4.

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2 Hypercube graphs

Graphs constructed on vertices of the d-dimensional cube $\{0,1\}^d$ provide important examples in geometric graph theory. Frankl and Wilson's proof that

$$\chi\left(\mathbb{R}^d\right) \geq \exp(Cd)$$

for some constant $C > 0$, for example, uses such graphs [2]. See also Kahn and Kalai's subsequent counterexample to Borsuk's conjecture [3].

Define the hypercube graph $C(d,u)$ to have vertices $V = \{0,1\}^d$, with two vertices adjacent if their Hamming distance is u. Note that Hamming distance u corresponds to Euclidean distance $\sqrt{u}$. So by dilating Euclidean space by a factor of $1/\sqrt{u}$, we see that $C(d,u)$ is a unit-distance graph in $\mathbb{R}^d$.

For d odd $C(d,u)$ is bipartite, so for our purposes these are not very interesting. For d even $C(d,u)$ has two isomorphic connected components — we denote one of these "half-cube" connected components on

2^{d-1} vertices by $H(d,u)$. In general we have that

$$\chi(R^d) \geq \chi(C(d,u)) = \chi(H(d,u)).$$

For example $C(5,2)$ is a graph on 32 vertices, regular of degree 10, and the half-cube $H(5,2)$ is a graph on 16 vertices.

Recall that for a graph G with $|V(G)|$ vertices and independence number $\alpha(G)$, we have that

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

One checks that the independence number $\alpha(H(5,2)) = 2$. Then the independence-number bound gives that $\chi(H) \geq 8$, and since H is a unit-distance graph in $\mathbb{R}^5$, this also gives that $\chi(\mathbb{R}^5) \geq 8$. This half-cube example is well known; see for example [5].

In Figure 1 we summarize our results on chromatic numbers of hypercube graphs for small d and u. In some cases we were able to compute $\chi(C(d,u))$ exactly. In some other cases we were not, but we were still able to compute the independence number $\alpha(C(d,u))$, giving a lower bound.

We obtain new lower bounds on $\chi(\mathbb{R}^d)$ for $d = 10$ and $d = 11$ this way, in both cases with Hamming distance $u = 4$. According to a calculation in Sage, the halfcube $H(10,4)$ has independence number $\alpha(H(10,4)) = 20$. There are 512 vertices, so then

$$\chi\left(H(10,4)\right) \geq \frac{512}{20} = 25.6,$$

and since the chromatic number is an integer,

$$\chi\left(\mathbb{R}^{10}\right) \geq 26.$$

According to another calculation in Sage, the half-cube $H(11,4)$ has independence number $\alpha(H(10,4)) = 32$. There are 1024 vertices, and then we have

$$\chi\left(R^{11}\right) \geq \frac{1024}{32} = 32.$$

This last example also gives a new record lower bound $\chi(\mathbb{R}^{12}) \geq 32$.

Since Hamming distance 4 corresponds to Euclidean distance 2, these graphs still have rational coordinates when rescaled to unit-distance graphs.

u \ d	2	3	4	5	6	7	8	9	10	11
6	1	1	1	1	2	4	≥ 5	≥ 7	≥ 11	≥ 16
4	1	1	2	4	7	8	8	≥ 15	≥ 26	≥ 32
2	2	4	4	8	8	8	8	≥ 13	≥ 13	≥ 13

Figure 1: Chromatic numbers of some hypercube graphs $\chi(C(d,u))$ for some small d and u. We restrict to u even, since $C(d,u)$ is bipartite for u odd. In some cases we compute chromatic number exactly, and have an independence-number lower bound. For $(d,u) = (10,2)$ and $(11,2)$ the lower bound is by monotonicity of rows.

3 Hyperplane slices of hypercube graphs

Let $C(d,u,s)$ denote the intersection of $C(d,u)$ with the hyperplane at height s

$$x_1 + \cdots + x_d = s.$$

Clearly, $C(d,u,s)$ is a unit distance graph in R^{d-1}, so

$$\chi\left(R^{d-1}\right) \geq \chi\left(C(d,u,s)\right).$$

For example, $C(10,4)$ has $2^{10} = 1024$ vertices and is regular of degree $\binom{10}{4}$, where the subgraph $C(10,4,5)$ has $\binom{10}{5} = 252$ vertices and is regular of degree 100.

We found with a computation in Sage that

$$\alpha\left(C(10,4,5)\right) = 12,$$

so

$$\chi\left(R^9\right) \geq \frac{252}{12} = 21.$$

This gives an alternate proof for the currently best known bound of Kupavskii and Raigorodskii [4].

4 A unit-distance graph in $\mathbb{R}^8$

Let G_0 be a graph whose vertices are the 240 shortest vectors in the $E8$ lattice. These may also be described as the vertices of an 8-dimensional Gosset polytope.

Concretely, there are 112 vertices with integer entries obtained from arbitrary permutations of the vectors

$$(\pm 2, \pm 2, 0, 0, 0, 0, 0, 0),$$

and 128 vertices with integer entries obtained from all vectors

$$(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$$

with an even number of minus signs.

Adjacency is with respect to Euclidean distance 4. So for example, $(2, 2, 0, 0, 0, 0, 0, 0)$ is adjacent to $(0, 0, 2, 2, 0, 0, 0, 0)$.

The graph G_0 lives on a 7-dimensional sphere of radius $2\sqrt{2}$. A pair of vertices on this sphere is adjacent if and only if the corresponding vectors are orthogonal.

Now define P to be the set of integer points within distance 4 of the origin.

Our algorithm is as follows.

1. Initialize G_0 as above.
2. Choose a vertex $x \in P$ outside of the current graph G_i.
3. If the independence number remains unchanged on adding vertex x to the graph G_i then add X, i.e. if $\alpha(G_i + x) = \alpha(G_i)$, then set $V(G_{i+1}) := V(G_i) + x$.
4. If there are any more points in P outside of the graph $V(G_i)$, go to step (2).

After experimentation, we found that at least 49 points (Figure 2) could be added without increasing the independence number, which then gives a unit-distance graph G with

$$\chi(G) \geq \frac{289}{16} = 18.0625,$$

- (-2, 0, -2, 0, 0, 2, 0, 2)
- (-2, 0, 0, 0, 2, 0, -2, 2)
- (-2, 0, 0, 2, 2, 2, 0, 0)
- (-2, 0, 2, 0, 0, 2, 0, 2)
- (-2, 0, 2, 2, 0, 0, 0, 2)
- (-2, 2, 0, 0, -2, 2, 0, 0)
- (-2, 2, 0, 0, 0, 0, 2, 2)
- (-2, 2, 2, 0, 2, 0, 0, 0)
- (0, -2, -2, 0, 0, 2, 0, 2)
- (0, -2, -2, 0, 2, 0, 0, 2)
- (0, -2, 0, 2, 2, 0, 0, 2)
- (0, 0, -2, 0, 2, 0, -2, 2)
- (0, 0, -2, 0, 2, 2, 2, 0)
- (0, 0, -2, 2, 2, 0, 2, 0)
- (0, 0, -1, 1, -1, 0, 0, 1)
- (0, 0, 0, 2, 0, 2, 2, 2)
- (0, 0, 0, 2, 2, 0, -2, 2)
- (0, 0, 2, 0, -2, 0, 2, 2)
- (0, 0, 2, 0, 2, -2, -2, 0)
- (0, 0, 2, 2, 2, 0, 2, 0)
- (0, 2, 0, 0, -2, -2, 0, 2)
- (0, 2, 0, 0, 2, 0, -2, -2)
- (0, 2, 0, 2, 0, 2, 0, -2)
- (0, 2, 0, 2, 2, 2, 0, 0)
- (1, -1, 1, -1, 1, 1, -1, 3)
- (1, 0, 0, 1, 1, 1, 0, 0)
- (1, 0, 1, 0, 1, 1, 0, 0)
- (1, 1, 0, 1, 1, 0, 0, 0)
- (1, 1, 1, 0, 0, 0, 0, 1)
- (1, 1, 1, 1, 0, 0, 0, 0)
- (2, -2, -2, 0, 0, 0, 2, 0)
- (2, -2, -2, 0, 0, 2, 0, 0)
- (2, -2, 0, -2, 0, 0, 2, 0)
- (2, -2, 0, 0, -2, 0, 0, 2)
- (2, -2, 0, 0, 2, 0, -2, 0)
- (2, 0, 0, 0, 0, -2, 2, -2)
- (2, 0, 0, 0, 0, 2, 2, -2)
- (2, 0, 0, 0, 2, -2, 0, -2)
- (2, 0, 0, 0, 2, 2, 0, -2)
- (2, 0, 0, 0, 2, 2, 2, 0)
- (2, 0, 0, 2, 0, -2, 2, 0)
- (2, 0, 0, 2, 2, 0, 0, -2)
- (2, 0, 0, 2, 2, 0, 0, 2)
- (2, 0, 2, -2, -2, 0, 0, 0)
- (2, 0, 2, 0, 0, 0, 2, -2)
- (2, 0, 2, 2, 0, 0, 0, 2)
- (2, 2, 0, 0, 0, 0, 2, 2)
- (3, -1, 1, -1, 1, -1, 1, 1)
- (3, -1, 1, 1, -1, -1, -1, -1)

Figure 2: List of 49 additional points added to the vertices of the 8-dimensional Gosset polytope to obtain a unit-distance graph with chromatic number at least 19.

so $\chi(\mathbb{R}^8) \geq 19$. Our graph G can be rescaled to have rational coordinates and unit distance, so this also gives the new lower bound $\chi(\mathbb{Q}^8) \geq 19$.

Mann used extensive computer-aided calculations to hunt for unit-distance graphs with large chromatic number in 2003 [6] — he established the lower bounds $\chi(\mathbb{Q}^6) \geq 10$, $\chi(\mathbb{Q}^7) \geq 13$, and $\chi(\mathbb{Q}^8) \geq 16$.

Cibulka studied similar constructions in [1]. He also added points to Gosset polytopes one at a time, and used computer-aided calculations to establish the bounds $\chi(\mathbb{Q}^5) \geq 8$ and $\chi(\mathbb{Q}^7) \geq 15$.

Unit-distance graphs built on Gosset polytopes go back at least to the landmark paper of Larman and Rogers in 1972 [5]. They in turn thanked McMullen for suggesting the idea. Larman and Rogers exhibited a configuration of 64 points, a "spindle" over the 7-dimensional Gosset polytope, to establish the previous record lower bound

$$\chi(\mathbb{R}^8) \geq 16.$$

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