POINTS IN A TRIANGLE FORCING SMALL TRIANGLES

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ABSTRACT. An old theorem of Alexander Soifer's is the following: Given five points in a triangle of unit area, there must exist some three of them which form a triangle of area $1/4$ or less. It is easy to check that this is not true if "five" is replaced by "four", but can the theorem be improved in any other way? We discuss in this article two different extensions of the original result.

First, we allow the value of "small" $1/4$ to vary. In particular, our main result is to show that given five points in a triangle of unit area, then there must exist some three of them determining a triangle of area $6/25$ or less.

Second, we put bounds on the minimum number of small triangles determined by n points in a triangle, and make a conjecture about the asymptotic right answer as $n \to \infty$.

1. INTRODUCTION

This article is in honor of Alexander "Sasha" Soifer's sixtieth birthday. I have known Sasha for about twenty years now; I first met him when I competed in his Colorado Math Olympiad in 1988. I remember that he teased me about looking even younger than I was (and also about my poor grades in school), but when we talked about mathematics he always spoke to me as his peer. We could speak a common language and share our curiosity and passion for solving problems.

Professor Soifer has solved many outstanding math problems, but posed many more. He has composed most of the problems for twenty-five (and counting) Colorado Math Olympiads, and written numerous research articles and five books about open problems in combinatorics and geometry. Perhaps the most important thing that I learned from him is that we are free to ask our own mathematical questions and pursue them as far as we are able. This was a very empowering idea to me when I was young.

Sasha has nice taste in problems too, and tells a good mathematical story [3]; I have sometimes felt a little haunted by problems which I first heard from him. One problem I have always remembered for its simple and elegant statement (and somewhat tricky proofs) is the following, which appeared on the Colorado Math Olympiad in 1988 [4]. It stayed with me, and I have occasionally sat down trying to see a new solution to it.
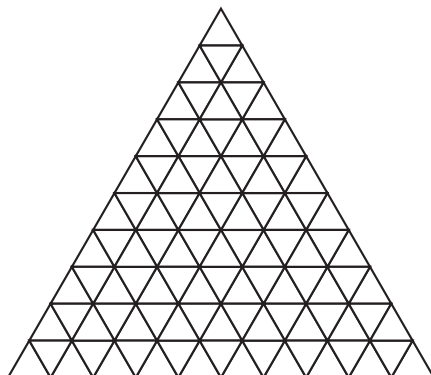
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FIGURE 1. Partitioning T into 100 congruent triangles.

Theorem 1.1. *Given five points in a triangle of unit area, there must exist some three of them forming a triangle of area $1/4$ or less.*

If the reader has never seen this problem, they are warmly invited to try it. The problem becomes easier if "five" is replaced by "six" or "seven," but it seems to take some work to get "five". Solutions by Alexander Soifer, Royce Peng, and Cecil Rosseau all appear in the book, "How does one cut a triangle?" which will soon be reprinted by Springer [1]. One wonders if this is the best possible result, and in a sense it is: it is easy to check that "five" can not be replaced by "four". But what about the $1/4$?

I must admit that I did not make any real progress toward proving Theorem 1.1 during the Olympiad, but now I will have a (very) small revenge on the problem by improving $1/4 = 0.25$ to $6/25 = 0.24$. In the spirit of Sasha Soifer and his wonderful journal Geombinatorics, we leave the reader with some open problems as well.

2. VARYING THE DEFINITION OF SMALL

Our main result is the following.

Theorem 2.1. *Given five points in a triangle of unit area, then some three of them form a triangle of area less than or equal to $6/25$.*

Since ratios of areas are preserved under affine transformation, we assume without loss of generality that our unit triangle T is equilateral. Let L denote the length of one side of T (i.e. $L = 2/\sqrt[4]{3}$). We partition T into 100 congruent equilateral triangles of side length $L/10$, as in Figure 1. Label the five points by $P = \{p_1, p_2, \ldots, p_5\}$.

We use the word "small" to refer to triangles of area $\leq 6/25$ for the rest of the section.

Lemma 2.2. *If for any pair of points in $p_i, p_j \in P$, $d(p_i, p_j) \leq 4L/10$ then three of the points in S form a small triangle. (In particular, if $d(p_1, p_2) < 4L/10$ then either $\{p_1, p_2, p_k\}$ forms a small triangle for some $k \in \{3, 4, 5\}$, or else $\{p_3, p_4, p_5\}$ forms a small triangle.)*

Proof of Lemma 2.2. For any three points x, y, z let $A(x, y, z)$ denote the area of the triangle they form. The locus of points x such that $A(p_i, p_j, x) \leq 24/100$ is a closed infinite strip S centered on line $p_i p_j$, and of width at least

$$2(\sqrt{3}/2)(6L/10) = (3\sqrt{3}/5)L > L.$$

Since the diameter of T is L, the intersection I of the complement of the strip with T, has at most one component. The main point is that if a third point is inside S we are done, and otherwise all three points lie in I, and we need only check that this forces the area of the triangle they form to be less than $24/100$.

By moving p_i and p_j along line $p_i p_j$ until they intersect the boundary of T if necessary, we only increase the width of the strip S, so we may assume without loss of generality that p_i and p_j are both on the boundary of T. If p_i and p_j are on the same side of T then I is a triangle with maximal area when $d(p_i, p_j) = 4L/10$, and this area is only $(4/10)^2 = 16/100$, in which case we are done, so we may assume that p_i and p_j are on two different sides of T.

Now suppose that $d(p_i, p_j) < 4L/10$. Then we can replace p_i and p_j by p'_i and p'_j, respectively, so that the line $p'_i p'_j$ is parallel to $p_i p_j$, and $d(p'_i, p'_j) = 4L/10$. This results in a new strip S' and denote the intersection of S' with T by I'. We claim that I' strictly contains I, as follows. Suppose that the distance between line $p_i p_j$ and $p'_i p'_j$ is Δ. The strip S is parallel to the strip S', and the width w' of strip S' is related to the width w of S by

$$(4L/10)w' = d(p_i, p_j)w.$$

We also have by basic trigonometry that

$$4L/10 - d(p_i, p_j) = \Delta \left(\frac{1}{\tan A} + \frac{1}{\tan 2\pi/3 - A} \right),$$

where A is the angle that line $p_i p_j$ makes with the sides of T, at p_j, as in Figure 2.

The minimum of the function $f(x) = 1/\tan(x) + 1/\tan(2\pi/3 - x)$ on the interval $x \in [0, 2\pi/3]$ is attained when $x = \pi/3$ and $f(x) = 2/\sqrt{3} > 1$. Now
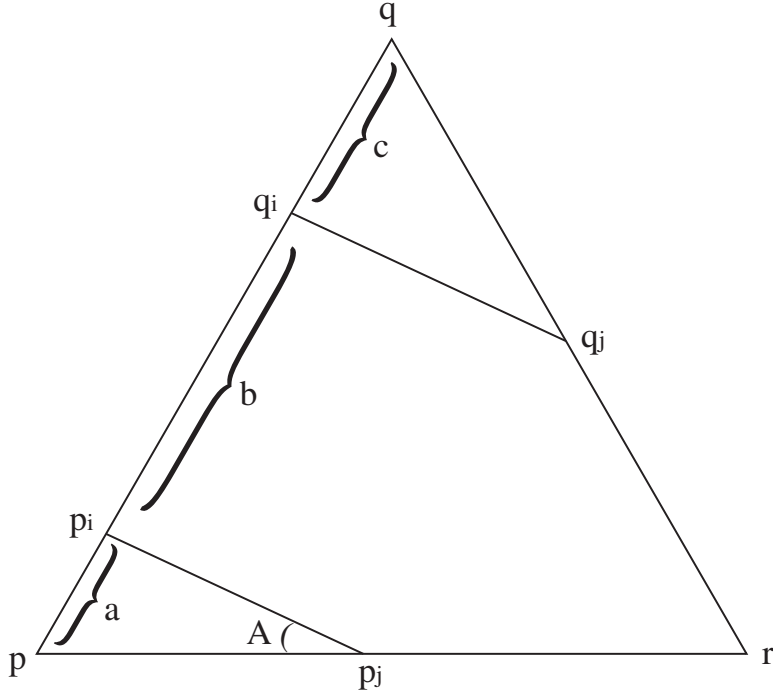
FIGURE 2. Proof of Lemma 2.2.

$$
\begin{aligned}
w - w' &= w\left(1 - \frac{d(p_i, p_j)}{4L/10}\right) \\
&= w\left(\frac{4L/10 - d(p_i, p_j)}{4L/10}\right) \\
&\geq \frac{w}{4L/10}(2/\sqrt{3})\Delta \\
&\geq \Delta,
\end{aligned}
$$

since $w \geq (3\sqrt{3}/10)L > 4L/10$.

This implies that I' contains I, and we may assume that $d(p_i, p_j) = 4L/10$.

Now label the corners of T by p, q, and r, so that p_i lies on side pr and p_j on side pq, as in Figure 2. If $d(p, p_i)$ and $d(p, p_j)$ are both greater than $4L/10$ then $d(p_i, p_j) > 4L/10$, and if they are both less than $4L/10$ then $d(p_i, p_j) < 4L/10$, but we are assuming that $d(p_i, p_j) = 4L/10$. So by symmetry, we assume without loss of generality that $d(p, p_i) \leq 4L/10$ and $d(p, p_j) \geq 4L/10$. In particular that $0 \leq A \leq \pi/3$.

Let q_i and q_j denote the other two intersections of the boundary of S with the boundary of T. Note that line $q_i q_j$ intersects side qr, and in particular I is a triangle. Let $a = d(p, p_i)$. Then by the Law of Sines,

$$a = \sin A (4L/10)\sqrt{3}/2 = \frac{4\sin A}{5\sqrt{3}}L.$$

Denote $b = d(p_i, q_i)$ and $c = d(q_i, r)$, so in particular $a + b + c = L$. We compute b in terms of A by

$$b = \frac{3\sqrt{3}/10}{\cos\left(A - \pi/6\right)},$$

so

$$c = L\left(1 - \frac{4}{5\sqrt{3}}\sin A - \frac{3\sqrt{3}}{10\cos\left(A - \pi/6\right)}\right).$$

The angles of triangle $q_i q_j q$ are $\pi/3$, $\pi/3 - A$, and $\pi/3 + A$, and applying the Law of Sines and a familiar formula for area of a triangle, its area is given by

$$|I| = \frac{1}{2}c^2\frac{\sin\left(\pi/3\right)\sin\left(\pi/3 + A\right)}{\sin\left(\pi/3 - A\right)}.$$

It seems that this is probably a monotone decreasing function of A on the interval $A \in [0, 2\pi/3]$, but we did not care to take derivatives of this function to prove it. So we instead take a slightly more indirect approach, and one that does not involve calculus. The idea is to start with the fact that c cannot be too large, which forces A large, which in turn forces c even smaller, etc., and we quickly reach a contradiction to the supposition that $|I| \geq 24/100$.

Since $0 \leq A \leq 2\pi/3$, we have $c \leq 4L/10$, so $c^2 \leq 16L^2/100$. Then if $|I| \geq 24/100$,

$$\frac{\sin\left(\pi/3\right)\sin\left(\pi/3 + A\right)}{2\sin\left(\pi/3 - A\right)} \geq \frac{3}{2L^2},$$

so

$$\frac{\sin\left(\pi/3 + A\right)}{\sin\left(\pi/3 - A\right)} \geq \frac{3}{2}.$$

Now $\sin\left(\pi/3 + A\right) \leq 1$, so this implies in particular that $\sin\left(\pi/3 - A\right) < 2/3$, and then since $0 \leq A \leq \pi/3$ this gives that $0.317 < A$. Plugging back

in to the formula for c gives

$$
\begin{aligned}
c &= L\left(1 - \frac{4}{5\sqrt{3}}\sin A - \frac{3\sqrt{3}}{10\cos(A - \pi/6)}\right) \\
&\leq L\left(1 - \frac{4}{5\sqrt{3}}\sin A - \frac{3\sqrt{3}}{10}\right) \\
&\leq L\left(1 - \frac{4}{5\sqrt{3}}\sin 0.317 - \frac{3\sqrt{3}}{10}\right) \\
&< 0.337L
\end{aligned}
$$

We repeat the argument from before. If $c < 0.337L$ and $|I| \geq 24/100$, then $\sin(\pi/3 - A) \leq c^2/0.24 < 0.474$, so $A > 0.553$. This gives in turn that

$$
\begin{aligned}
c &\leq L\left(1 - \frac{4}{5\sqrt{3}}\sin 0.553 - \frac{3\sqrt{3}}{10}\right) \\
&< (24/100)L.
\end{aligned}
$$

a contradiction to the assumption that $|I| > 24/100$, since

$$
|I| = cd(r, q_j)/L^2 < c/L.
$$

$\square$

Then we immediately have the following.

Lemma 2.3. *If any two points of P lie in a triangle of side length $4L/10$, or a hexagon of diameter $4L/10$, then some three points form a small triangle.*

These are many such triangles and hexagons suggested by our partition of T, as in Figure 3.

We further restrict the possible arrangements of points by using the fact that a triangle in a parallelogram of area A has area at most $A/2$. (This observation is used in Soifer's original proof of Theorem 1.1.)

Lemma 2.4. *If any three points of P lie in a parallelogram of area $A \leq 48/100$, then they form a triangle of area no more than $A/2 \leq 24/100$.*

We make use of three such kinds of parallelograms, illustrated in Figure 4, which we will call $(2, 10)-$, $(3, 8)-$, and $(4, 6)-$ parallelograms.

We consider three cases.

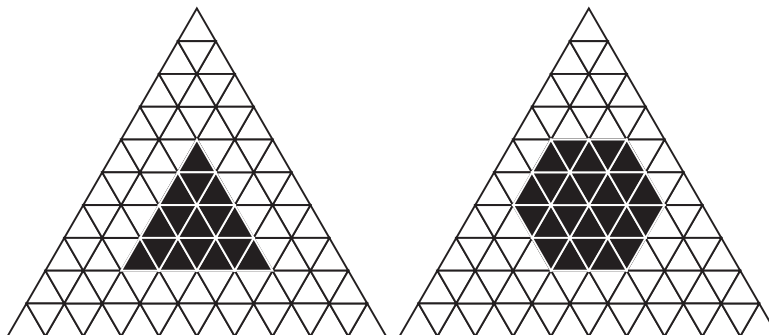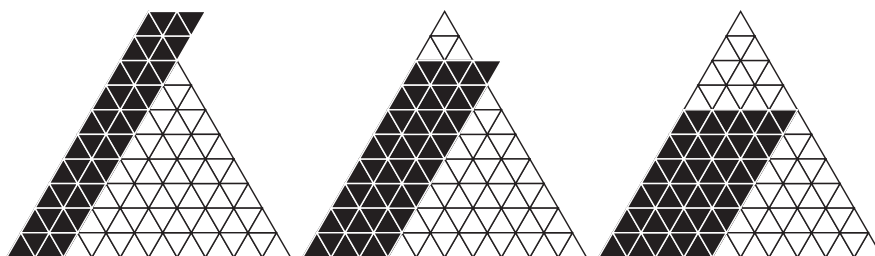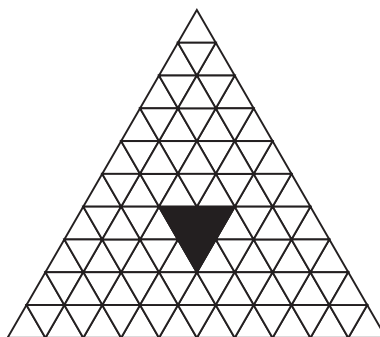Case I is that there is a point of P inside the central shaded triangle in Figure 5.

FIGURE 3. A triangle and hexagon of diameter $4L/10$.

FIGURE 4. Parallelograms of area $A \leq 48/100$. From left to right, $(2, 10)-$, $(3, 8)-$, and $(4, 6)-$ parallelograms.

FIGURE 5. Case I: At least one point is in the shaded triangle.

Assuming Case I, by the hexagon case of Lemma 2.3, there can be no other points in the wider shaded region in Figure 6, or else we are already done. But that leaves at least four points in the three corner white triangles. Two of them must lie in the same triangle, but then by the triangle case of Lemma 2.3 we are done.

Case II is that there is a point p_1 in the shaded region in Figure 7. By symmetry, assume that p_1 is in the upper triangle. There are no other points
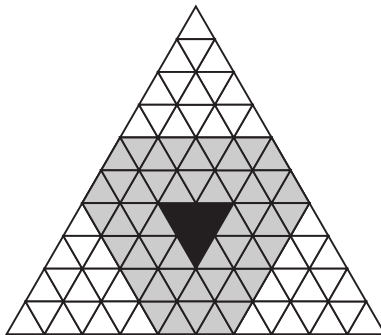
FIGURE 6. Case I: There can be no more than the given point in the greater shaded region. Hence four points lie in the three white triangles.
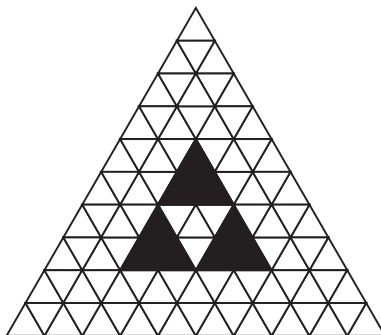

FIGURE 7. Case II: At least one point p_1 lies in one of the three shaded triangles. Assume that p_1 in the top triangle.

in the wider shaded region of Figure 8 or we are done, by the hexagon case of Lemma 2.3. There is at most one point in the top white triangle, and so there are at least three points in the lower white region. The case of no points in the top triangle and four in the bottom is easy to dispense with by using Lemma 2.3. So assume that there is one point, p_2, in the top triangle.

Now $p_3, p_4,$ and p_5 are all in the lower white region of Figure 7. They can not all be in the lower two strips, or we are done by the parallelogram lemma with a $(2, 10)$-parallelogram, so there is at least one point p_3 on the third or fourth row, by symmetry assume it is in the triangle on the left, as in Figure 9.

By Lemma 2.4 with a $(3, 9)$-parallelogram along the left edge of T and containing points p_2 and p_3, p_1 must be in the bottom right corner triangle of side length $L/10$, as in Figure 11. Then applying Lemma 2.4 with $(3, 8)-$ and $(4, 6)$-parallelograms gives that p_4 and p_5 are in the white region at the bottom of Figure 11.
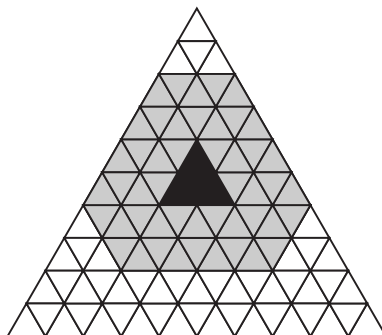
FIGURE 8. Case II: There can be no more than the given point in the greater shaded region, and there is at most one point in the top white triangle. So at least three points lie in the white region at the bottom.
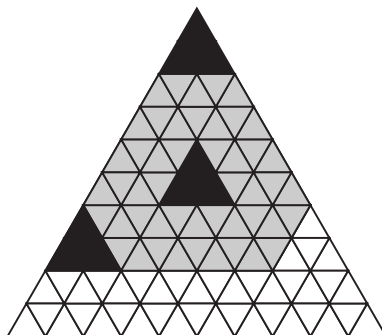

FIGURE 9. Case II: p_1, p_2, and p_3 are in the three shaded triangles: p_1 in the center, p_2 at the top, and p_3 on the bottom-left.
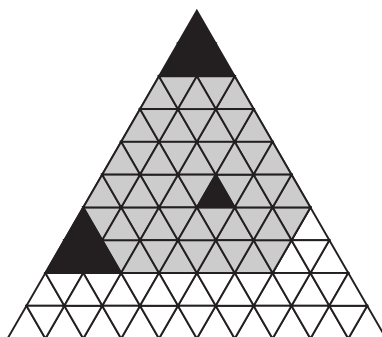

FIGURE 10. Case II: Then by the parallelogram lemma, with a $(3, 9)$-parallelogram, point p_1 is in the bottom right corner of the center triangle, or we are done.
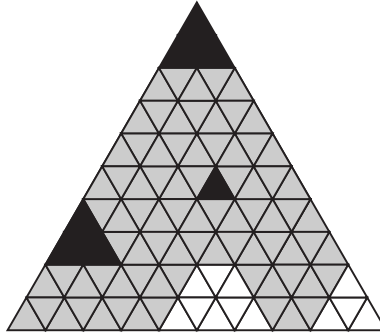
FIGURE 11. Case II: Applying Lemma 2.4 with $(3, 8)-$ and $(4, 6)-$ parallelograms gives than p_4 and p_5 are in the white regions at the bottom of T. Then p_3, p_4, and p_5 form a small triangle.
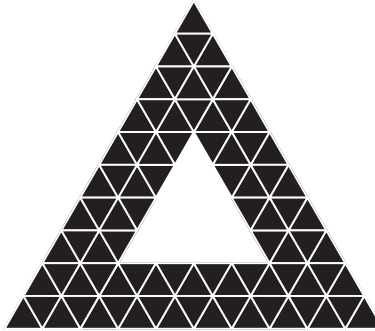

FIGURE 12. Case III: All five points lie in the shaded area.

We claim that then p_3, p_4, and p_5 form a triangle of area $24/100$ or less. Given any particular location of p_4 and p_5, triangle $p_3p_4p_5$ attains its maximum value (given p_4 and p_5) with p_3 at a corner of its boundary triangle. If p_3 is one of the bottom two corners, then the three points are in a $(2, 10)$-parallelogram, and if it is the top corner, they are in a $(3, 8)$-parallelogram. In either case, the area of the triangle is less than $24/100$ by Lemma 2.4.

Then Case III is the only possibility left, that all five points contained in the outer shaded region in Figure 12.

We break the outer strip into three diamond and trapezoid shaped regions, Figure 13.

Any pair of the trapezoids can be covered with a hexagon and two triangles of diameter $4L/10$, so by Lemma 2.3, they contain a total of at most three points from S. It is quick to deduce from this, that the points are distributed $2 - 1 - 1$ in the trapezoids, with one point in the diamond opposite the trapezoid with two points, or we are done. By symmetry we assume p_1
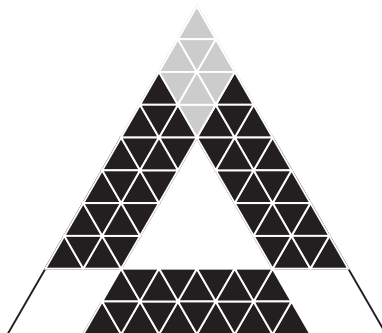
FIGURE 13. Case III: We partition the outer strip into six smaller regions, diamonds and trapezoids. Lemma 2.3 places severe restrictions on where the points may lie, if they are to form no small triangles. In fact without loss of generality we assume that p_1 is in the diamond on top, p_2 and p_3 in the trapezoids on the left and right, and p_4 and p_5 in the bottom trapezoid.
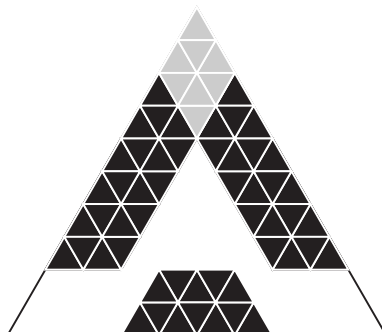

FIGURE 14. Case III: $(3, 8)$-parallelograms force p_4 and p_5 into in an even smaller trapezoid, shaded at bottom, but then we are done by Lemma 2.3.

is in the diamond on top, p_2 and p_3 are in the trapezoids on the left and right, respectively, and p_4 and p_5 are in the bottom trapezoid.

By $(3, 8)$-parallelograms at the top vertex of T and going down the left and right sides, p_4 and p_5 are in an even smaller trapezoid, as seen in Figure 14, but then we are done by Lemma 2.3.

This completes the proof of the main result, but begs the question:

Question 2.5. *What is the minimum s of all σ such that among every five points in a triangle of unit area, some three of them form a triangle of area*
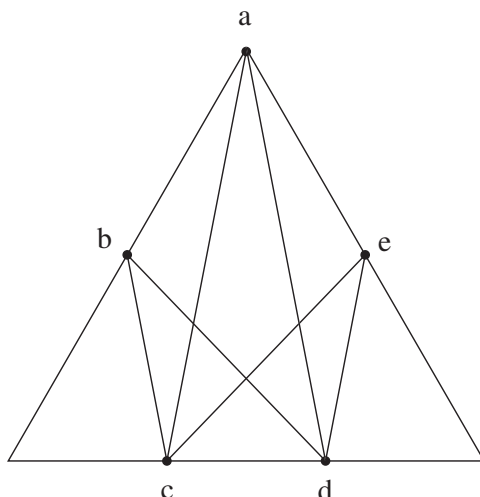
FIGURE 15. Five points in a unit area triangle, with none of the triangles having area less than $1/6$. Point a is at the corner, b and e are on midpoints, and c and d partition the bottom side into three equal segments. Note that triangles abc, ade, bcd, and cde all have area equal to $1/6$.

less than or equal to σ? (Note that s is actually a minimum and not just an infinium, by compactness. [2])

By Figure 15, we have $s \geq 1/6$. This might be best possible.

Conjecture 2.6. $s = 1/6$.

3. MANY SMALL TRIANGLES

In this section we return to using "small" to refer to triangles of area $\leq 1/4$**.** There is another way in which Theorem 1.1 might be improved, besides varying the $1/4$. Is it possible that given five points in a triangle of unit area, they must form not only one small triangle, but more than one? We conjecture that this is the case; in particular we believe the following holds.

Conjecture 3.1. *Given five points in a triangle of unit area, then they form at least three small triangles.*

This is not true with "three" replaced by "four", by the arrangement in Figure 16. If the conjecture holds then the following corollary would follow, by averaging: Given $n \geq 5$ points in a triangle of unit area, there they form
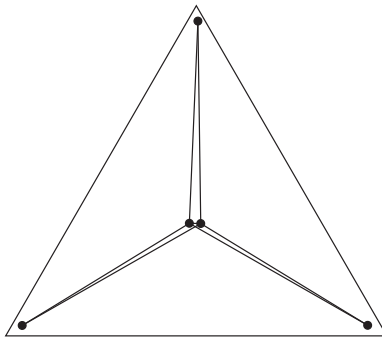
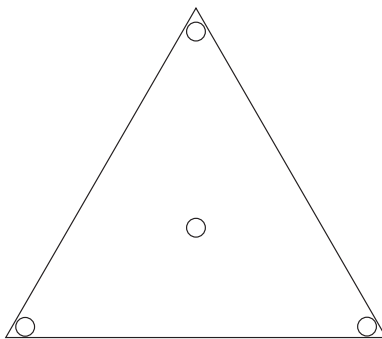FIGURE 16. There are only three small triangles.

FIGURE 17. If there approximately $n/4$ points in each of the small circles, then there are $4(n/4)^3 = n^3/16 \approx (3/8)\binom{n}{3}$ big triangles, hence approximately $(5/8)\binom{n}{3}$ small triangles.

at least

$$\frac{3}{\binom{5}{2}}\binom{n}{3} = \frac{3}{10}\binom{n}{3}$$

small triangles. This leads us to the following question.

Question 3.2. *What is the supremum A of all $\alpha \in [0,1]$ such that given (sufficiently large) n points in a triangle of unit area, they form at least*

$$\alpha\binom{n}{3}$$

small triangles?

We observe that $A \leq 5/8$, by the construction in Figure 17, and conjecture that asymptotically this arrangement gives the smallest number of small triangles.

Conjecture 3.3. $A = 5/8$.

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